Mathematics Methods for Early Childhood

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JANET STRAMEL

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Welcome to Early Childhood Mathematics! This course satisfies the Early Childhood Unified requirements in the state of Kansas for a teaching license Birth to Grade 3.

Most people agree that early childhood includes the period from infancy until eight years of age, characterized by rapid and complex growth in physical, cognitive, and social domains. Math skills must be taught in early childhood. Children should be provided a foundation to succeed in elementary school and beyond. Teachers should focus lessons in early childhood around the basic skills that will help to advance future mathematics. From preschool to the end of elementary school, children are setting the foundation for future life skills.

Learning mathematics is "a 'natural' and developmentally appropriate activity for young children" (Ginsberg, Lee, and Boyd, 2008). Through their everyday interactions with the world, many children develop informal concepts about space, quantity, size, patterns, and operations. Unfortunately, not all children have the same opportunities to build these informal and foundational concepts of mathematics in their day-to-day lives (Sherman-LeVos, 2010).

Young children are naturally curious, and the best time to begin mathematics is at a time while the young child's brain is rapidly developing. Mathematics in early childhood helps children develop critical thinking and reasoning skills early on *and* it's the key to the foundation for success in their formal schooling years.

ACKNOWLEDGEMENTS

Thank you to my husband, Dean Stramel, for supporting me during this project, as well as serving as my editor. From reading the early drafts to giving me feedback and advice, he was critical in the completion of this project. His encouragement and support sustains me!

Thank you to Laney Roth for giving me critical feedback. Your friendship, both professionally and personally, is invaluable as we support early childhood mathematics teachers.

DEDICATION

For Tracy and Joshua

CHAPTER 1

The Importance of Early Childhood Mathematics

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"Mathematical knowledge begins during infancy and undergoes extensive development over the first 5 years of life. It is just as natural

for young children to think mathematically as it is for them to use language, because "humans are born with a fundamental sense of quantity" (Geary, 1994, p. 1), as well as spatial sense, a propensity to search for patterns, and so forth" (Clements, Sarama, & DiBiase, 2004).

Emergent mathematics is the earliest phase of development of mathematical and spatial concepts. Emergent mathematics encompasses the skills and attitudes that a child develops in relation to math concepts throughout the early childhood period. Emergent mathematics and those foundational math skills are not optional. They are necessary to the success of each and every student.

It is widely accepted that literacy learning begins the day a child is born. Reading to infants and toddlers and preschoolers is an early predictor of positive literacy success. Mathematical understanding can be regarded in the same way. During the first months of life, a child begins to construct the foundations for mathematical concepts. For example, before a child can count, he/she can construct ideas about mathematics. They recognize "more," "less," and basic equality.

Read the articles, "Early Math Skills Predict Later Academic Success" by Nancy Christensen. You will notice that early childhood mathematics is critical for the future success of children (Duncan, 2007). As you read this article, reflect on what you have learned, what difference it makes to you as a future teacher and the children you will teach, and what you can do with this information.

Best Teaching Practices

As a teacher of mathematics, please remember these fundamental beliefs.

1. Thinking about the problem, not just the answer, is what is most important. The teacher should be the facilitator,

not the one with all the answers.

2. Process is more important than product. Mathematics is not just "facts" to be memorized. The concepts children learn are of upmost importance; and then the memorization will come. Children must have experiences through play and using manipulatives and "reinvent" the concepts of mathematics in their own minds.

The National Council of Teachers of Mathematics (2014), in its publication Principles to Action, represents a unified vision of what is needed for teaching all students.

The eight Mathematics Teaching Practices can be found in the Executive Summary.

1. Establish mathematics goals to focus learning.
2. Implement tasks that promote reasoning and problem solving.
3. Use and connect mathematical representations.
4. Facilitate meaningful mathematical discourse.
5. Pose purposeful questions.
6. Build procedural fluency from conceptual understanding.
7. Support productive struggle in learning mathematics.
8. Elicit and use evidence of student thinking.

Read more about the Mathematics Teaching Practice in the Executive Summary.

Treat Children as Mathematicians

- Mathematicians often work for an extended time on a single problem. Allow your students ample time to work on a problem.

- Mathematicians collaborate with their colleagues and study the work of others. Allow your students to collaborate, argue, consult, defend, ask, explain, and pose questions to others.
- Mathematicians must prove for themselves that their solution is correct. Children must also prove to themselves that they have the correct solution. Give students time to think and discuss. Don't be too quick to tell a student their answer is correct or incorrect; instead ask the student to explain their answer.
- The problems mathematicians work on are complex. Good problems ask students to find innovative solutions without a time limit. Problems should spark children's thinking, rather than promote rote memorization.
- Mathematicians get satisfaction from the problem-solving process and take pride in their solutions. Children will understand the concepts and procedures more if they are allowed to use their own thinking processes. This also allows children to make connections to prior knowledge and real-life experiences.
- Mathematicians use unsuccessful attempts as stepping-stones to solutions. We need to emphasize mathematical thinking rather than just getting the correct answer.

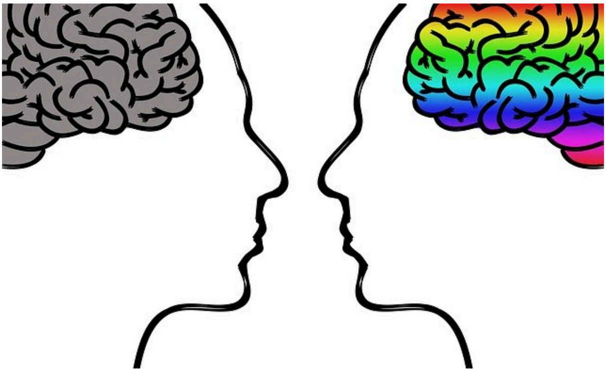
NCTM/NAEYC Position Statement on Early Start in Learning Mathematics

The National Council of Teachers of Mathematics (NCTM) and the National Association for the Education of Young Children (NAEYC) published a Joint Position Statement (2010) affirming high-quality, challenging, and accessible mathematics education for 3- to 6-year-old children is a vital foundation for future mathematics learning.

The ten research-based, essential recommendations to guide your classroom practice are:

1. Enhance children's natural interest in using mathematics to make sense of their world.
2. Build on children's background experience and knowledge.
3. Base instruction on knowledge of children's cognitive, linguistic, physical, and social-emotional development.
4. Strengthen children's problem solving and reasoning processes as well as representing, communicating, and connecting mathematical ideas.
5. Ensure a coherent curriculum that is compatible with sequences of important mathematical ideas.
6. Provide for children's deep and sustained interactions with mathematical ideas.
7. Integrate mathematics with other activities and other activities with mathematics.
8. Provide ample time, materials, and teacher support for children to engage in play where they explore mathematical ideas.
9. Use a range of experiences and teaching strategies to introduce mathematical concepts, methods, and language.
10. Support children's learning by thoughtful and continuous assessment.

Allow children to work for long periods of time on one problem. Good problems can have multiple solutions and different ways to arrive at those solutions; therefore, allow children to use their own methods for solving a given problem. Children's excitement about mathematics comes from their own thinking abilities; encourage social interaction that promotes children to act as young mathematicians by requiring them to prove their answer and all the steps they took to attain the answer. Additionally, allow children to be wrong many times before being right. It is important to encourage children to see "wrong" answers as a natural part of mathematical processes.



Mindsets

"It's not that I'm so smart, it's just that I stay with problems longer." – Albert Einstein

Mindset is a person's usual attitude or mental state. For example, if you have an environmentalist mindset, you probably bring your own bags to the grocery store. The noun mindset was first used in the 1930s to mean "habits of mind formed by previous experience."

In a **growth mindset**, people believe that their most basic abilities can be developed through dedication and hard work—brains and talent are just the starting point. This view creates a love of learning and a resilience that is essential for great accomplishment. Virtually all great people have had these qualities.

Fixed Mindset vs. Growth Mindset

In a fixed mindset, people believe their qualities are fixed traits and therefore cannot change. According to Dweck (2007), when a student has a fixed mindset, they believe that their basic abilities, intelligence, and talents are fixed traits. They think that you are born with a certain amount and that's all you have.

Jo Boaler says that "it is through mistakes that knowledge will grow. Teachers do not need to be the all knowing expert. It is better to model being a curious inquirer. Someone who seeks to play with knowledge, learn with students and model being OK making mistakes and learning from them" (Boaler, 2016)

Read the following articles for more information about Growth Mindset:

- Decades of Scientific Research that Started a Growth Mindset Revolution from Mindset Works
- When Teachers Think Differently About Themselves As Math Learners, Students Benefit by Krysten Crawford
- Developing Mathematical Mindsets: The Need to Interact with Numbers Flexibly and Conceptually by Jo Boaler
- Watch the video "Boosting Math" by Dr. Jo Boaler and her students.

Want to know more about Growth Mindset? Check out these resources:

The following websites have more information on Growth Mindset.

- The Kansas State Department of Education (KSDE) has resources on the Mathematics Page
- Becoming a Math Person
- The Power of Belief – Mindset and Success
- Building a Mathematical Mindset

Community

- Five Videos to Explore Mathematical Mindset
- We're All Board with Mathematical Abilities (And Why That's Important)
- 8 Teaching Habits that Block Productive Struggle in Math Students
- "The Knowledge" – The World's Toughest Taxi Test

Social and Emotional Learning

In the Kansas Mathematics Standards (2017), there is a reference to the “Rose Capacities and Kansas Social, Emotional, and Character Development Model Standards.” Mandated by the Kansas State Legislature, the Rose Capacities are the intersection between cognitive and non-cognitive skills, which support the Standards for Mathematical Practice.

A whole child approach is critical to ensure that every student is healthy, safe, engaged, supported, and challenged. The Kansas Mathematics Standards provide a problem-solving approach that engages students in a conceptual understanding of mathematics. The Standards for Mathematical Practice ensure that students solve real-world problems by working with peers; formulating, communicating, and critiquing arguments; and persevering through difficulty. As students internalize these mathematical practices, they engage interpersonal and intrapersonal skills, also known as social and emotional learning (SEL) competencies.

To read more about SEL in Mathematics, look at the Inside Mathematics website.

CHAPTER 2

Mathematics Milestones

JANET STRAMEL

Children grow faster in the first two years of life; more than at any other period of development. During the first two years, the brain grows from 25% of its adult weight to 75% of its final adult weight. Additionally, gross motor skills develop; they use large muscle groups to crawl, sit, stand, and lift things. Fine motor skills develop more slowly: stacking blocks, stringing beads, zipping zippers, and buttoning buttons.

Children are learning about the world around them through experiencing it through sight, smell, sound, touch, and taste. They are also learning by thinking about things, forming hypotheses, testing them, and problem solving. This is Piaget's sensorimotor stage of development. This stage is from birth to approximately 2 years of age, and is a period of rapid cognitive growth and development. **Object permanence** is the main development during the sensorimotor stage and is vital to mathematics because it is the understanding that an object continues to exist even if it is not seen. According to Geist, "object permanence is the first step toward representation. When a child knows something exists even when he can't see it, it means he has a mental concept of the object in the absence of its presence. This is one of the bases upon which more advanced mathematics will be built. Numbers are representative of objects and quantities even if we do not see them" (Geist, 2009).

At a very early age, babies use mathematics concepts to make sense of their world. For example, they can signal "more" when they want more food. In addition, they know the difference between familiar and unfamiliar adults which is "sorting and classifying." They have "spatial reasoning" as they play with boxes of different sizes, and they use "patterns" as they say words and phrases.

As early as **6 months old**, children begin to develop an understanding of "more." They ask for "more milk" or "more food." Babies from **0-12 months** begin to predict the sequence of events, such as running water means bath time, they begin to classify things into certain categories, such as toys that make noise and toys that don't, and they begin to understand relative size, such as parents are big and babies are small (Morin, 2014).

At **12 to 18 months old**, children start to recognize patterns and understand shapes as they play with objects. They begin to sort familiar objects by characteristics, and they enjoy filling and emptying containers. Many children at this age can complete simple puzzles when the puzzle pieces show the whole object.

Although each child progresses at different levels, generally children of **18 months to 24 months old** can show “one” and “two” on their fingers. They know some number words. Although they do not understand quantity, they do begin to understand “take one” or “give me one.” Children can match same-sized shapes. Puzzles and toys of squares, triangles, and circles help children develop spatial reasoning.

At **24 to 30 months old**, children are learning important mathematical concepts and skills through their play. They recognize patterns and begin to use logical reasoning to solve everyday problems. They can sort shapes and stack blocks by size, and be able to distinguish between “big and small” and “fast and slow.” Children will be able to say some numbers, although they may skip some numbers in the sequence. For example, they may say, “one, two, three, five, ten…” By the age of 3, children should be able to count one to ten. Children at this stage of development generally understand addition and subtraction with the numbers one and two.

At approximately **three years old**, a child may attend a preschool. But what does mathematics teaching and learning look like in a preschool classroom? All children have the potential to learn mathematics that is challenging and at a high-level (Sarama & Clements, 2009). There is a wide range of development between ages three and five. There will be children who are “almost potty trained” as well as children who have begun learning to read. “Sit-and-listen” approach is only effective if it is highly engaging and exciting, for example, a story being read to the class. Therefore, teaching should consist almost entirely of “active learning.” Teaching should emphasize student-directed/teacher-facilitated activities. Worksheets are inappropriate at this level.

At **3-4 years old**, children can count up to 30; and be able to count backwards from ten. Additionally, they can use **ordinal numbers** "first," "second," and "third." At this age, children will be able to use superlatives such as "big, bigger, biggest." At age 4, children will understand the concept of addition as putting together and subtraction as taking from when objects are in front of them. They also recognize shapes, begin to sort things by color, shape, size, or purpose, understand that numerals stand for numbers, and have the spatial awareness to complete puzzles (Morin, 2014).

At **age four**, children begin to have **one-to-one correspondence** in which numbers correspond to specific quantities. For example, a child would point to the number of blocks on the table and say, "1, 2, 3, 4, 5. I have five blocks."

At **age five**, children can identify the larger of two numbers and recognize numerals up to 20. They can copy or draw shapes, and understand basic time concepts such as days of the week. By **age six**, children can add and subtract within ten mentally and solve simple word problems.

Read more about Mathematical Milestones: Children's Development of Mathematical Concepts: Ages 0-4 (Infants, Toddlers, & Preschoolers).

Infants and Toddlers

Mathematics is all around us! And we use mathematical language all the time, even without realizing it. For example, when we sort laundry into colors and whites, we are sorting and classifying.

Keeping score at a game is number and operations. Spatial relationships occur when we give someone directions, and when we make comparisons such as big and little, that is measurement. Everyday activities include patterns and order. Teachers must make mathematics visible to children through math talks (Greenberg, 2012).

Infants and toddlers understand **number and operations** as we talk with them about numbers, quantity, and one-to-one correspondence. According to Greenberg (2012), we can say the following to young children:

- You have two eyes, and so does your bear. Let's count: 1-2.
- I have *more* crackers than you do. See, I have 1, 2, 3, and you have 1, 2. I'm going to eat one of my crackers. Now I have the *same* as you!
- That's the third time you have said mama. You've said mama three times!

Infants and toddlers understand **geometry** by recognizing and naming shapes. According to Greenberg (2012), we can talk with our children about understanding shapes and spatial relationships:

- Look, Samantha went *under* the stool and Hartley is on top.
- You are sitting *next* to your sister.
- Some of the cookies are *square* and some are *round*.

Algebraic thinking begins as children see patterns, relationships, and change as they recognize relationships that make up a pattern or create repetitions. For example, you can say to infants and toddlers:

- Grandpa has stripes on his shirt – red, white, blue, red, white, blue.

- Let's clap to the beat of this song.
- I put the blocks *in* the bucket; you dump them *out*. I put the blocks back *in* the bucket; you dump them *out*.

Data analysis begins as infants and toddlers gather, sort, classify, and analyze information to make sense of their world. Teachers can talk with children about data by saying the following:

- Let's put the *big* lid on the *big* bowl.
- You *always* smile when mommy sings.

Adapted from Greenberg, 2012

Math talks begin as soon as a child is born. During bath time, diaper time, meal time, or on walks or shopping trips is the perfect time to talk about shapes and sizes, talk about patterns, and describe things that are different. Mathematics comprehension is more than just numbers. Math is also about sorting things into categories such as big, bigger, and biggest; solving problems using patterns; solving spatial problems; and using logic.

Read more about Math Talks with infants and toddlers in this article, More, All Gone, Empty, Full: Math Talk Every Day in Every Way.

Kindergarten and First Grade

As children enter kindergarten and first grade, there are many changes in the physical, cognitive, and social-emotional development. Children in kindergarten can sit and attend to a formal lesson for approximately 10 minutes. At this level, the focus of mathematics instruction should be through experiences. By first grade, children can sit for a 15-20 minute formal lesson and although they are developing selective attention (the ability to decide what to attend to and what to ignore), the focus should still be on mathematics teaching through experiences.

Children in kindergarten and first grade are developing their gross and fine motor skills. According to Thelen and Smith, the development of coordination comes from physical play such as running and jumping and skipping (Thelen & Smith, 1996). Fine motor skills are developed as children write numbers and letters and use a pencil.

Kindergarteners can add by counting on their fingers, and start with six on the other hand; they can copy or draw symmetrical shapes, and can follow multi-step directions using first and next. In first grade, students can predict what comes next in a pattern, know the difference between two- and three-dimensional shapes, count to 100 by ones, twos, fives, and tens, and can write and recognize numbers, and do basic addition and subtraction up to 20 (Morin, 2014).

The Value of Play

Play does not guarantee mathematical development, but it offers rich possibilities. As the teacher, it is up to you to observe and develop a classroom when children have many opportunities to discover mathematical concepts through play. Douglas Clements says, “Anyone who is pushing arithmetic onto preschoolers is wrong. Do not hurry children.” (Clements, 2001, pg. 270). Mathematics should be about joy and challenges, not pressure and compulsion.

For example, children begin to recognize mathematical patterns by listening to music. They naturally make relationships between different-size and different-shaped blocks by sorting. According to Clements (2001), “Good early mathematics is broader and deeper than mere practice in counting and adding. It includes debating which child is bigger and drawing maps to ‘treasure’ buried outside. Quality mathematics instruction includes providing loads of unit blocks, along with loads of time to use them; asking children to get just enough pencils for everyone in the group; and challenging children to estimate and check how many steps are required to walk to the playground.”

Classrooms that offer children opportunities to engage in play-based learning activities help children grow not only academically. “Play is a particular attitude or approach to materials, behaviors, and ideas and not the materials or activities or ideas themselves; play is a special mode of thinking and doing” (McLane, 2003, p. 11). As discussed in an earlier chapter, there is a strong connection between early mathematics learning and later success. One determining factor of later achievement is quality early mathematical experiences.

Play is critical to a young child’s learning. Mathematics activities must be developmentally appropriate by “integrating logical thinking, decision making comparison, and the making of relationships throughout the child’s day into everything they do”

(Geist, 2009). This is the goal of emergent mathematics – to immerse children in mathematics, number, and number concepts. Mathematics at this level must be taught through experiences. Engaging young children in play-based learning, teachers can include the following:

- Rhythm and music allows children to recognize mathematical patterns,
- Blocks and shapes gives children opportunities to make relationships between them, such as same and different,
- Everyday activities such as counting help children develop number concepts,
- Manipulatives such as stringing beads, multicolored balls, blocks, etc. encourage children in imaginative play,
- Everyday routines and common activities such as snack time or distributing plates and napkins, and
- Math games encourage children to make mathematical relationships.

Adapted from Geist, 2009

Read the article, "Mathematics in the Preschool" by Douglas H. Clements for more information about quality early mathematical experiences.

Watch the video, "The Decline of Play," by Peter Gray.

Play is how children learn! Mathematics games, fantasy play, puzzles, manipulatives, and games allow children to learn and grow.

CHAPTER 3

Mathematics in Preschool

Preschool children are very active, and any program and teacher must take that into account. Emotionally, preschoolers are inquisitive and explorative. Cognitively, they are in the preoperational stage of development in which they begin to engage in symbolic play and learn to manipulate symbols. However, Piaget noted that children at this stage do not yet understand concrete logic. Children at this stage learn through pretend play.

During the first half of the preoperational stage, children are in the “symbolic function substage.” Children at this stage are generally two- to four-years old. They let one object stand in for another and use symbols and signs, such as numbers. They do this through pretend play; therefore, give your preschool children as much time as possible for imaginative play. This then leads to the “intuitive thought substage” in which children are not logical, but think intuitively. Children at this stage ask many questions and are very curious.

Douglas Clements (2001), suggests that we need preschool mathematics for four reasons: 1) Preschoolers experience mathematics at a basic level, and that needs to be improved, 2) Many children, especially those from minority backgrounds or underrepresented groups, have difficulty in school mathematics and therefore preschool teachers should address those equity issues, 3) Preschoolers do possess informal mathematical abilities and use mathematical ideas in real life, and preschool teachers should capitalize on their interests, and 4) brain research has shown that preschoolers’ brains undergo significant development, their experiences and learning affects the structure and organization of their brains, and preschoolers’ brains grow most as a result of complex activities (Clements, 2001).

Preschool mathematics can be divided into two groups: numerical and measurement. Numerical activities are discrete while measurement activities are continuous. Numerical concepts ask the question, “How many?” and are referred to as discrete quantities because they can be counted.

Mathematics during the preschool years should focus on number, geometry, measurement, algebra and patterns, and problem-solving. At age three, children can hold up a number of fingers to indicate a quantity.

Number

Rote counting is the ability to say the numbers in order and involves the memorization of numbers; meaningful or **rational counting** is the ability to assign a number to the objects counted. Children at age three can hold up fingers to indicate a quantity and by age four can count to five or ten, and can tell you what comes next. **Cardinal numbers** say how many of something there are, such as one, two, three, four, five; and they answer the question "How many?" **Ordinal numbers** tell the position of something in the list, such as first, second, third, fourth, fifth, etc.

There are "rules" for writing and saying number words; it is the base-ten number system that we use (10 digits, 0-9) and we place the greater value on the left. For example, three hundred sixty-two is written as 362. According to Seo and Ginsburg (2004), a child's ability to write and say numbers does not guarantee their application. Just because a child can say numbers does not mean that they know the quantity associated with that number. Therefore, preschool children begin to put their understanding of "one" to use as they "count up" and develop the meaning of adding one more.

Preschool children must develop the concepts of order and seriation. **Order** is the ability to count a number of objects once and only once. **Seriation** is the process of putting objects in a series, for example from smallest to largest. Additionally, young children begin to group objects by their characteristics, such as yellow and blue.

Geometry

Preschool children can use directional words such as "up and down," and "over and under" as well as comparing words such as "bigger and smaller," or "longer and shorter." Additionally, children at ages three to four recognize and name shapes. Naming a shape is mathematics, it is language arts. Mathematics comes in as students recognize and classify the attributes of those shapes. Additionally, students are beginning to compose and decompose shapes. For example, they may be able to make a square with two triangles.

Measurement

Three-year olds can lay two objects side-by-side and tell which one is longer. By age four, children begin to use non-standard units to measure things. For example, they can tell you how many shoes long a desk is, although they need to use many shoes. They are not yet ready to use one shoe repeatedly.

Children in preschool do not learn to tell time, but they are learning the concept of time. They talk about yesterday, today, and tomorrow.

Algebra and Patterns

Preschool children do algebra by recreating patterns and making their own patterns. Children can recognize, describe, extend, and create patterns from a simple repeating pattern such as "red, blue, red, blue" to a more complex pattern such as "red, red, blue, red, red, blue." They also notice growing patterns such as "1, 2, 3, 4" or "2, 4, 6, 8."

Problem-Solving

Problem solving is critical at all levels. Allow students to solve a problem without stepping in too quickly. Children begin to link words and concepts; therefore, teachers can begin to use story problems for teaching mathematics. In kindergarten, the words should be simple and short and by first grade, students begin to write their strategy in the problem-solving process.

Each of these content areas will be further developed in subsequent chapters.

Critical Mathematics Concepts for Preschool Children

Activities in the preschool classroom must incorporate the use of manipulatives and hands-on learning, and the main emphasis should be on number sense. Do not use worksheets or independent practice in the preschool classroom, instead plan activities that will develop a strong sense of number and patterns. The following domains are critical concepts as you teach preschool children to be mathematically proficient:

Counting and Cardinality

Number sense is the foundation for success in mathematics and is the first vital skill for preschool children (Resilient Educator, 2021). The ability to count accurately is a part of number sense, but also to see the relationship between numbers, such as addition and subtraction. Children should be able to demonstrate simple counting skills before kindergarten. This includes counting to 20, ordering numbers, identifying how many are in a set without counting (subitizing), and understanding that the quantity does not change regardless of the arrangement of the items. Additionally, preschool children should understand cardinality, in which the last number said is the number of items in the set.

Operations and Algebraic Thinking

Mathematical ideas become "real" when teachers and students use words, pictures, symbols, and objects. Young children are naturally visual and can build those relationships between numbers and the item represented; therefore, teachers must use pictures and objects to clarify that relationship. As children learn to count, they will learn that the number symbol represents the number of items shown.

Patterns are things that repeat in a logical way. Manipulatives can help children sort, count, and see patterns. An AB pattern means that two items alternate, such as red, blue, red, blue, red, etc. ABC patterns means that three items are in the pattern, such as bear, cow, giraffe, bear, cow, giraffe, etc. Students will learn to make predictions about what would come next in a pattern.

Number and Operations in Base 10

Preschool children begin to understand that the number "ten" is made up of "ten ones," although this is a difficult concept. Teachers should allow children to count on their fingers one to ten.

Showing students the meaning of the words more, less, bigger, smaller, more than, and less than can help young children understand estimation.

Measurement and Data

Finding length, height, and weight using inches, feet, pounds, or non-standard units is measurement. Also in this skill area is measurement of time. Teachers should ask their students to notice objects in their world and compare them, for example, “The stepstool is bigger than the chair. Do you think it will fit under the chair?”

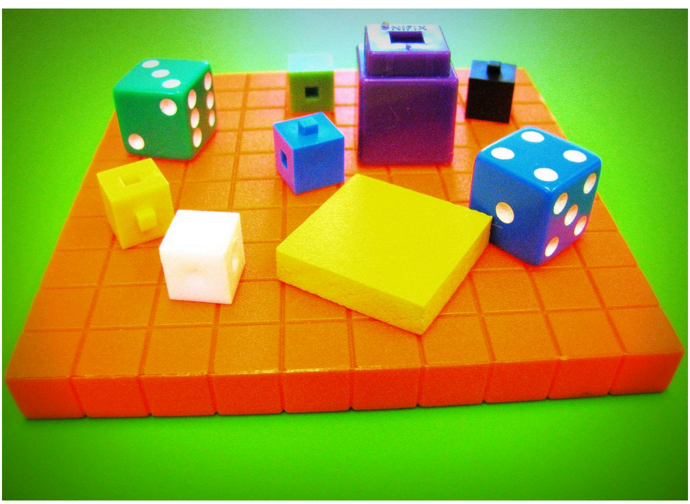
Sorting is a skill that preschool children should do often. One way to sort is by color; another way is by another attribute. Teachers can ask students to count the toys in a basket, and then sort them based on size, color, or their purpose. Check out this excerpt from the book Exploring Math and Science in Preschool by the National Association for the Education of Young Children – Sorting Activities for Preschoolers by William C. Ritz.

Geometry

Spatial sense is geometry, but at the preschool level it is the ability to recognize shape, size, space, position, direction, and movement. Teachers can talk with children about shapes – count the sides or describe the shape. Furthermore, talk with children about shapes in their world, such as “The pizza is round,” or “The sandwich is a rectangle.”

Calendar Time

Morning calendar time is a daily part of many preschool classrooms. There is a ritual when children sit on the floor and talk about today, look at yesterday, find out about tomorrow, and write out the date. Understanding that time is sequential is critical for young children. They think about before and after, later and earlier, and future and past events. According to Beneke, Ostrosky, and Katz (2008), preschool children generally cannot judge distances or lengths of time. For example, they do not understand that a field trip is in five days and differently than if it is in eight days. And it is different for young children to judge units of time. And although a true understanding of calendar dates comes with maturity, using the calendar to teach other concepts is also valuable time spent in the classroom. For example, vocabulary (month, year, weekend), sequencing (yesterday, today, and tomorrow), and patterns (Monday, Tuesday, Wednesday). They also begin to recognize numbers. Additionally, teachers can use calendar time to teach social skills, colors, letters, and integrate science as they talk about the weather (Beneke, Ostrosky, & Katz, 2008).



Manipulatives

Manipulatives are the mainstay of a preschool mathematics classroom (Geist, 2009). Math **manipulatives** are physical objects that are designed to represent explicitly and concretely mathematical ideas (Moyer, 2001). Students need time to explore and manipulate materials in order to learn the mathematics concept. According to Carol Copple (2004), children should be given many opportunities to manipulate a wide variety of things and teachers should provide children to "mess about."

One productive belief from the NCTM publication, Principles to Action (2014), states, "Students at all grade levels can benefit from the use of physical and virtual manipulative materials to provide visual models of a range of mathematical ideas." Students at all grade levels can benefit from manipulatives, but especially at the elementary level. Using manipulatives can

- provide your students a bridge between the concrete and abstract.
- serve as models that support students' thinking.
- provide another representation.
- support student engagement.
- give students ownership of their own learning.

Adapted from "The Top 5 Reasons for Using Manipulatives in the Classroom."

Everyday activities can be used to promote mathematics. For example, during snack time children divide up snacks, count plates, and notice the one-to-one correspondence between the number of children and the number of napkins needed.

CHAPTER 4

Mathematics Standards

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Standards describe what students are expected to learn in each grade and each subject. Each state Department of Education creates standards for schools within the state. These standards become the basis for the way teachers are trained, what they teach and what is on state standardized tests that students take. Without standards, districts and schools don't have goals. By matching what is taught in the classroom to the standards in each subject area, students will know what teachers should be teaching, what students should be learning, and what they will be tested on (GreatSchools Staff, 2012).

As an early childhood teacher, you need to be familiar with three sets of standards: the Kansas Early Learning Standards, the Kansas Mathematics Standards, and the Standards for Mathematical Practice.

Kansas Early Learning Standards

The Kansas Early Learning Standards were first completed in 2006 with a second revision done in 2009 and updated in 2014. The standards are aligned with the K-12 College and Career Ready Standards and provide information and guidance to early childhood providers and teachers, including early primary grade teachers, on the developmental sequence of learning for children from birth through kindergarten promoting continuity between early childhood years and the primary grades (K-3). It is a dynamic resource that providers and teachers will be able to use as they plan activities for and engage in conversations with young children and their families around early learning.

The Kansas Early Learning Standards are "statements describing the skills and knowledge that young children, ages birth through five, should know and be able to do as a result of participating in high quality early childhood programs. This knowledge and ability provides the foundation for future success in kindergarten and later in life. Such continuity can facilitate smooth transitions and clarify communication between programs both vertically and horizontally. This alignment with K-12 supports school readiness as well as school success" (KSDE, 2014).

Kansas Early Learning – Mathematics Knowledge

Research shows that young children can do mathematics and solve problems. Long before entering school, young children explore and use mathematics and do it naturally. Children at play begin to learn essential math skills such as counting, equality, addition, subtraction, estimation, planning, patterns, classification and measurement. They compare, notice similarities and differences, and group toys and materials. This ability to organize information into categories, quantify data and solve problems helps children learn about time, space and numbers. Over time, they develop the vocabulary and skills to:

- Measure,
- Describe patterns,
- Express order and position, and
- Describe relationships between objects in the environment.

Mathematics helps children make sense of the physical and social worlds around them and they naturally incorporate math as they make comments such as:

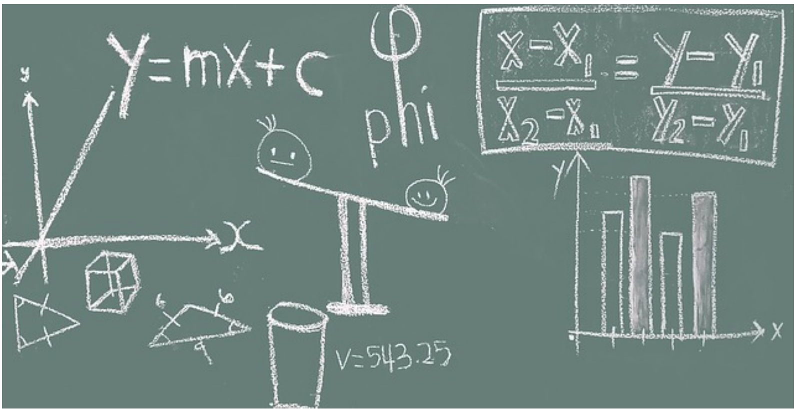
- "He has more than I do!"
- "That won't fit in there, it's too big."
- "I can't move it, it's too heavy."

Organization of the Kansas Early Learning Standards

The Kansas Early Learning Standards – Mathematics Knowledge are categorized into four domains:

1. Counting and Cardinality
2. Operations and Algebraic Thinking
3. Measurement and Data
4. Geometry

The mathematics content is divided by age levels: Young Infant (by 8 months), Mobile Infant (by 18 months), Toddler (by 36 months); Pre 3 (by 48 months), Pre 4 (by 60 months), and Kindergarten (by the end of K).



Kansas Mathematics Standards

The **Kansas Mathematics Standards** were adopted by the State Board of Education in August 2017. These Standards define what students should understand and be able to do in mathematics. Asking a student to understand something means asking a teacher to assess whether the student has understood it. But what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from. The Standards set grade-specific standards but do not define the methods or materials necessary to support students.

The Kansas Mathematics Standards provide information on what students should know and be able to do at different grade levels in mathematics. These standards are guidelines school districts can use to develop their mathematics curriculum. **They are not the curriculum.** In Kansas, each school district develops its own mathematics curriculum and teachers decide on how they will provide instruction to ensure student's learning of mathematics.

Organization of the Kansas Mathematics Standards

There are two sets of standards; content standards and the Standards for Mathematical Practice.

The **standards for mathematical content** are focused on number and quantity, algebra, functions, geometry, modeling, and statistics and probability. For kindergarten through grade 8, the **content standards** are organized by grade levels. The 9-12 standards do not use grade bands, but instead have broad conceptual categories. All content standards are organized by Domain, Cluster, and then Standards.

Domains are larger groups of related standards. The Domain is the big idea.

Clusters are groups of related standards. At the beginning of each cluster, the bold label is referred to as the *Cluster Heading*

Standards define what *students* should understand and be able to do.

You can access the Kansas Mathematics Standards here.

Kansas Mathematics Standards K-8

Domains are descriptions of the mathematical content to be learned. These are elaborated through clusters and standards.

Clusters may appear in multiple grade levels in the K-8 Common Core. There is increasing development as the grade levels progress. Clusters begin with a content statement. For example, in grade 1, **"Extend the counting sequence."**

Standards are what students should know and be able to do at each grade level. The standards reflect both mathematical understandings and skills, which are equally important. For example, in grade 1, "Count to 120 (recognizing growth and repeating patterns), starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral." (1.NBT.1)

The Kansas Mathematics Standards document is intended to be a resource to teachers. The document is interactive: hyperlinks are in blue font and underlined and red words are roll-over words. Hovering over the word provides a definition of the term identified. In addition, the Learning Progressions are referenced throughout the document.

There is no substitute for reading the standards yourself. Read the Kansas Mathematics Standards.

Standards for Mathematical Practice

The **Standards for Mathematical Practice** (SMPs) describe varieties of expertise that mathematics teachers at all levels should seek to develop in their students. The SMPs represent the habits of mind that represent thinking and understanding.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.

3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

What do the Standards for Mathematical Practice Mean?

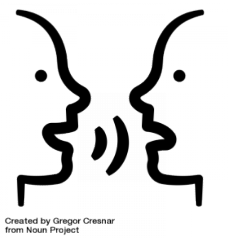
SMP1: *Make sense of problems and persevere in solving them* means to understand the problem, find a way to solve the problem, and work until it is done. Your students will basically be using this first SMP in every mathematics problem, every day.

In order to help your students "make sense of problems and persevere in solving them," allow wait time for yourself and your students. Work for progress and "aha" moments. Mathematics becomes about the process and not about the one right answer. Lead with questions, but don't pick up a pencil and do the problem for your students.

This means that sometimes your students will struggle. This is called "productive struggle," which is the process of effortful learning that develops grit and creative problem-solving. When students come across a problem they don't know how to solve, we don't want them to give up. Instead, we want our students to make connections to what they already know, and try different ways of solving a problem. Read more about Productive Struggle at the Renaissance website.

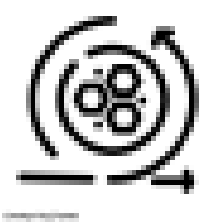
SMP2: *Reason abstractly and quantitatively* means to break apart a problem and show it symbolically, in pictures, or in any way other than the standard algorithm. This means that students can **contextualize** **decontextualize**
understand the relationships between problems and mathematical representations, as well as how the symbols represent strategies for solution. If students have a problem, they should be able to break it apart and show it symbolically, with pictures, or in any way other than the standard algorithm. Conversely, if students are working on a problem, they should be able to apply the math to the situation.

Help your students ***"reason abstractly and quantitatively"*** by asking your students to draw representations of problems. Be sure to have lots of manipulatives for your students to access. Or let your students decide what to do with data themselves instead of telling them the type of graph to use. Pose purposeful questions that will lead your students to a conceptual understanding. Or ask your students to draw their thinking, with and without traditional number sentences.



SMP3: *Construct viable arguments and critique the reasoning of others* means that your students will be able to talk about mathematics, using mathematical language and vocabulary to support or oppose the work of others.

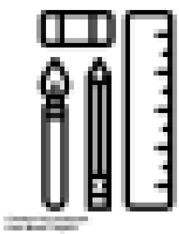
Teachers can help students *"construct viable arguments and critique the reasoning of others"* by posting mathematical vocabulary and expect students to use it. Encourage student discourse, and make sure that your classroom environment is a safe place for students to share and discuss ideas.



SMP4: *Model with mathematics* means to solve real-world problems, organize data, and understand the world around you. This SMP is often misinterpreted. It is not about manipulatives, but about the symbols and expressions that represent a situation.

Math limited to the math class is worthless. Help your students "model with mathematics" by having them use math in other classes, such as science, art, music, and reading. Include real graphs, articles, and current data to make the mathematics relevant and real, or have your students create real-world problems using their mathematical knowledge. You can promote student thinking by asking, "What is happening here?" and "How can I represent this with mathematical expressions?"

You can learn more about this SMP from Robert Kaplinsky's website.



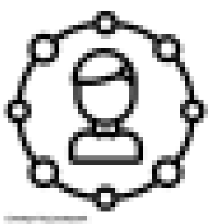
SMP5: *Use appropriate tools strategically* means students can select the appropriate math tools to use, and use that tool correctly to solve problems. Don't tell your students which tool to use; leave that decision open-ended and then discuss what worked best and why. Tools do not necessarily need to be so much about the real situation but rather about the mathematical situation presented.

Teachers can help their students ***"use appropriate tools strategically"*** by providing manipulatives to assist in developing students' mathematical thinking. Encourage your students to draw or use a model to identify how the problem responses may be alike or different.

The Sadlier Math Blog has some great ideas for manipulatives at the different grade levels.

SMP6: ***Attend to precision*** means that students speak and solve mathematics problems with exactness and precision. Encourage your students to use precise mathematical language. This SMP is more than just a correct solution; it is also about communicating clearly and precisely using mathematical language.

To help your students ***"attend to precision"*** ask them to use precise and exact mathematical language. Their measurements should be exact, their numbers should be precise, and their explanations must be detailed.



SMP7: ***Look for and make use of structure*** means to find patterns and repeated reasoning that can help students solve more complex problems. For young students, this could mean recognizing fact families, inverses, or mathematical properties.

"Use of structure" means . . . what is going on in the math that allows you to apply the equations or symbols or to draw the diagram you can draw. It is the outline of what's going on in the situation.

Help your students "look for and make use of structure" by asking them to identify multiple strategies and then select the best one. Repeatedly break apart numbers and problems into different parts.

Use what you know is true to solve a new problem. Prove solutions without relying on the algorithm. For example, my students are changing mixed numbers into improper fractions. They have to prove to me that they have the right answer without using the "steps."

This SMP applies what we know about general rules to specific situations . . . deductive reasoning (properties, definitions, characteristics of the math, relationships between the quantities, and explanations of why certain things are happening).

SMP8: *Look for and express regularity in repeated reasoning* means looking at the big picture while working on the details of the problem. This SMP extends our thinking beyond just a simple repeating pattern; it encourages students to ask, "Where does this repetition lead you in terms of making a generalization to other situations?"

One way to help your students ***"look for and express regularity in repeated reasoning"*** is to show students how the problem works. As soon as they "get it," ask them to generalize to a variety of problems. Don't ask your students to work 50 of the same problem; instead, ask them to take their mathematical reasoning and apply it to other situations.

The description of this SMP includes patterns but extends our thinking beyond just a simple repeating pattern or number pattern. It's a broader look at patterning; repetition in procedures and thinking. Encourage student reflective questioning. Ask, "Where does this repetition lead you in terms of making generalizations to other situations?" or "If we have many examples of something that are somehow the same, what can we learn from that?"

Find SMPs in the Kansas Mathematics Standards, pages 9–13, or in the blue box at the beginning of each grade level. Also note the summary of the SMPs and questions to ask your students that will help develop these habits of mind. In addition, you can find more about SMPs at the Common Core State Standards for Mathematics website.

Learning Progressions

Included with the Kansas Mathematics Standards document are references to the learning progressions. These progressions describe the progression of a mathematical topic across grade levels. In addition, they are linked throughout the standards to provide insight into the content standards.

All learning progressions can be found at the KSDE website.

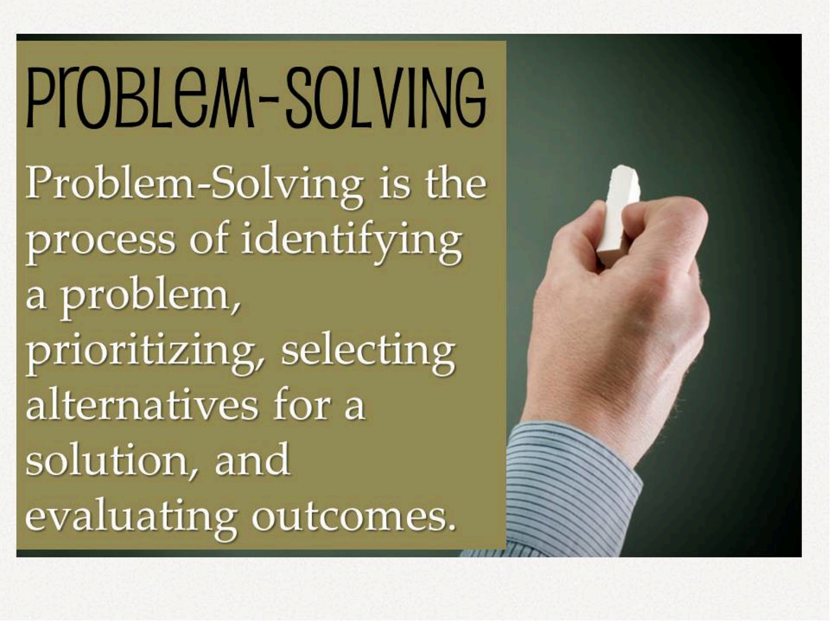
The Kansas Early Learning Standards, the Kansas Mathematics Standards, the Standards for Mathematical Practice, and the

Learning Progressions all provide clarity and specificity to the mathematics we teach. As you plan lessons, you will need to know the standards that define what students should know and be able to do at each grade level.

CHAPTER 5

Teaching Mathematics Through Problem Solving

JANET STRAMEL



In his book "How to Solve It," George Pólya (1945) said, *"One* *of the most important tasks of the teacher is to help his students. This task is not quite easy; it demands time, practice, devotion, and sound principles. The student should acquire as much experience of independent work as possible. But if he is left alone with his problem without any help, he may make no progress at all. If the teacher helps too much, nothing is left to the student. The teacher should help, but not too much and not too little, so that the student shall have a reasonable share of the work." (page 1)*

What is a **problem** in mathematics? A problem is "any task or activity for which the students have no prescribed or memorized rules or methods, nor is there a perception by students that there is a specific 'correct' solution method" (Hiebert, et. al., 1997). Problem solving in mathematics is one of the most important topics to teach; learning to problem solve helps students develop a sense of solving real-life problems and apply mathematics to real world situations. It is also used for a deeper understanding of mathematical concepts. Learning "math facts" is not enough; students must also learn how to use these facts to develop their thinking skills.

According to NCTM (2010), the term "problem solving" refers to mathematical tasks that have the potential to provide intellectual challenges for enhancing students' mathematical understanding and development. When you first hear "problem solving," what do you think about? Story problems or word problems? Story problems may be limited to and not "problematic" enough. For example, you may ask students to find the area of a rectangle, given the length and width. This type of problem is an exercise in computation and can be completed mindlessly without understanding the concept of area. **Worthwhile problems** includes problems that are truly problematic and have the potential to provide contexts for students' mathematical development.

There are three ways to solve problems: teaching for problem solving, teaching about problem solving, and teaching through problem solving.

Teaching *for* problem solving begins with learning a skill. For example, students are learning how to multiply a two-digit number by a one-digit number, and the story problems you select are multiplication problems. Be sure when you are teaching for problem solving, you select or develop tasks that can promote the development of mathematical understanding.

Teaching *about* problem solving begins with suggested strategies to solve a problem. For example, “draw a picture,” “make a table,” etc. You may see posters in teachers’ classrooms of the “Problem Solving Method” such as: 1) Read the problem, 2) Devise a plan, 3) Solve the problem, and 4) Check your work. There is little or no evidence that students’ problem-solving abilities are improved when teaching about problem solving. Students will see a word problem as a separate endeavor and focus on the steps to follow rather than the mathematics. In addition, students will tend to use trial and error instead of focusing on sense making.

Teaching *through* problem solving focuses students’ attention on ideas and sense making and develops mathematical practices. Teaching through problem solving also develops a student’s confidence and builds on their strengths. It allows for collaboration among students and engages students in their own learning.

Consider the following worthwhile-problem criteria developed by Lappan and Phillips (1998):

- The problem has important, useful mathematics embedded in it.
- The problem requires high-level thinking and problem solving.
- The problem contributes to the conceptual development of students.
- The problem creates an opportunity for the teacher to

assess what his or her students are learning and where they are experiencing difficulty.

- The problem can be approached by students in multiple ways using different solution strategies.
- The problem has various solutions or allows different decisions or positions to be taken and defended.
- The problem encourages student engagement and discourse.
- The problem connects to other important mathematical ideas.
- The problem promotes the skillful use of mathematics.
- The problem provides an opportunity to practice important skills.

Of course, not every problem will include all of the above. Sometimes, you will choose a problem because your students need an opportunity to practice a certain skill.

Key features of a good mathematics problem includes:

- It must begin where the students are mathematically.
- The feature of the problem must be the mathematics that students are to learn.
- It must require justifications and explanations for both answers and methods of solving.

Problem solving is not a neat and orderly process. Think about needlework. On the front side, it is neat and perfect and pretty.

But look at the back.

It is messy and full of knots and loops. Problem solving in mathematics is also like this and we need to help our students be "messy" with problem solving; they need to go through those knots and loops and learn how to solve problems with the teacher's guidance.

When you teach *through* **problem solving**, your students are focused on ideas and sense-making and they develop confidence in mathematics!

MATHEMATICS TASKS AND ACTIVITIES THAT PROMOTE TEACHING THROUGH PROBLEM SOLVING

Choosing the Right Task

Selecting activities and/or tasks is the most significant decision teachers make that will affect students' learning. Consider the following questions:

- How is the activity done?
  - Teachers must do the activity first. What is

problematic about the activity? What will you need to do BEFORE the activity and AFTER the activity? Additionally, think how your students would do the activity.

- What is the purpose of the activity?
    - What mathematical ideas will the activity develop? Are there connections to other related mathematics topics, or other content areas?
- Can the activity accomplish your learning objective/goals?

Low Floor High Ceiling Tasks

By definition, a “**low floor/high ceiling task**” is a mathematical activity where everyone in the group can begin and then work on at their own level of engagement. Low Floor High Ceiling Tasks are activities that everyone can begin and work on based on their own level, and have many possibilities for students to do more challenging mathematics. One gauge of knowing whether an activity is a Low Floor High Ceiling Task is when the work on the problems becomes more important than the answer itself, and leads to rich mathematical discourse [Hover: ways of representing, thinking, talking, agreeing, and disagreeing; the way ideas are exchanged and what the ideas entail; and as being shaped by the tasks in which students engage as well as by the nature of the learning environment].

The strengths of using Low Floor High Ceiling Tasks:

- Allows students to show what they can do, not what they can't.
- Provides differentiation to all students.
- Promotes a positive classroom environment.
- Advances a growth mindset in students
- Aligns with the Standards for Mathematical Practice

Examples of some Low Floor High Ceiling Tasks can be found at the following sites:

- YouCubed – under grades choose Low Floor High Ceiling
- NRICH Creating a Low Threshold High Ceiling Classroom
- Inside Mathematics Problems of the Month

Math in 3-Acts

Math in 3-Acts was developed by Dan Meyer to spark an interest in and engage students in thought-provoking mathematical inquiry. Math in 3-Acts is a whole-group mathematics task consisting of three distinct parts:

Act One is about noticing and wondering. The teacher shares with students an image, video, or other situation that is engaging and perplexing. Students then generate questions about the situation.

In **Act Two**, the teacher offers some information for the students to use as they find the solutions to the problem.

Act Three is the "reveal." Students share their thinking as well as their solutions.

"Math in 3 Acts" is a fun way to engage your students, there is a low entry point that gives students confidence, there are multiple paths to a solution, and it encourages students to work in groups to solve the problem. Some examples of Math in 3-Acts can be found at the following websites:

- Dan Meyer's Three-Act Math Tasks
- Graham Fletcher3-Act Tasks]
- Math in 3-Acts: Real World Math Problems to Make Math Contextual, Visual and Concrete

Number Talks

Number talks are brief, 5-15 minute discussions that focus on student solutions for a mental math computation problem. Students share their different mental math processes aloud while the teacher records their thinking visually on a chart or board. In addition, students learn from each other's strategies as they question, critique, or build on the strategies that are shared.. To use a "number talk," you would include the following steps:

1. The teacher presents a problem for students to solve mentally.
2. Provide adequate "**wait time**."
3. The teacher calls on a students and asks, "What were you thinking?" and "Explain your thinking."
4. For each student who volunteers to share their strategy,

write their thinking on the board. Make sure to accurately record their thinking; do not correct their responses.

5. Invite students to question each other about their strategies, compare and contrast the strategies, and ask for clarification about strategies that are confusing.

"Number Talks" can be used as an introduction, a warm up to a lesson, or an extension. Some examples of Number Talks can be found at the following websites:

- Inside Mathematics Number Talks
- Number Talks Build Numerical Reasoning

Saying "This is Easy"

"This is easy." Three little words that can have a big impact on students. What may be "easy" for one person, may be more "difficult" for someone else. And saying "this is easy" defeats the purpose of a growth mindset classroom, where students are comfortable making mistakes.

When the teacher says, "this is easy," students may think,

- "Everyone else understands and I don't. I can't do this!"
- Students may just give up and surrender the mathematics to their classmates.
- Students may shut down.

Instead, you and your students could say the following:

- "I think I can do this."
- "I have an idea I want to try."
- "I've seen this kind of problem before."

Tracy Zager wrote a short article, "This is easy": The Little Phrase That Causes Big Problems" that can give you more information. Read Tracy Zager's article here.

Using "Worksheets"

Do you want your students to memorize concepts, or do you want them to understand and apply the mathematics for different situations?

What is a "worksheet" in mathematics? It is a paper and pencil assignment when no other materials are used. A worksheet does not allow your students to use hands-on materials/manipulatives [Hover: physical objects that are used as teaching tools to engage students in the hands-on learning of mathematics]; and worksheets are many times "naked number" with no context. And a worksheet should not be used to enhance a hands-on activity.

Students need time to explore and manipulate materials in order to learn the mathematics concept. Worksheets are just a test of rote memory. Students need to develop those higher-order thinking skills, and worksheets will not allow them to do that.

One productive belief from the NCTM publication, Principles to Action (2014), states, “Students at all grade levels can benefit from the use of physical and virtual manipulative materials to provide visual models of a range of mathematical ideas.”

You may need an “activity sheet,” a “graphic organizer,” etc. as you plan your mathematics activities/lessons, but be sure to include hands-on manipulatives. Using manipulatives can

- Provide your students a bridge between the concrete and abstract
- Serve as models that support students’ thinking
- Provide another representation
- Support student engagement
- Give students ownership of their own learning.

Adapted from “The Top 5 Reasons for Using Manipulatives in the Classroom”.

CHAPTER 6

Early Number Concepts and Number Sense

JANET STRAMEL

As a teacher of mathematics, remember these fundamental beliefs as you develop early number concepts in your students.

1. Thinking about the problem, not just the answer, is what is most important.
2. The teacher should be the facilitator, not the one with all the answers.
3. Process is more important than product. Mathematics is not just "facts" to be memorized. The concepts children learn are of utmost importance; and then the memorization will come. Children must experience through play and using manipulatives and "reinvent" the concepts of mathematics in their own minds.

To help children develop early early number concepts, use activities that focus on **verbal counting**, **one-to-one correspondence**, **cardinality**, and **subitizing**.

Watch this video from Graham Fletcher on "The Progression of Early Number & Counting."

Verbal counting is critical in developing quantitative thinking in young children. A focus on counting in context is one way to build a child's mathematical abilities, therefore it is critical to count with children every day. Some activities include:

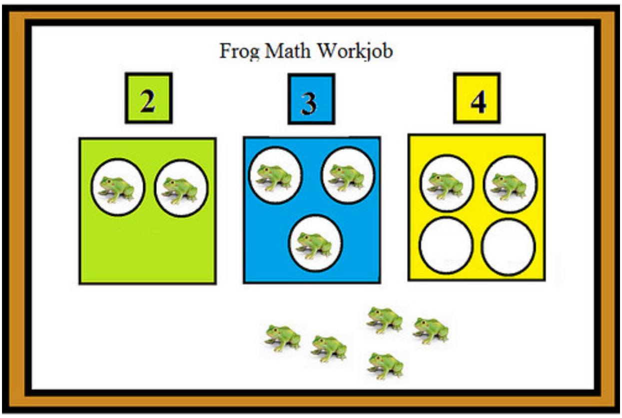
- Practice counting words and written numerals with pictures or representations of objects.
- Count aloud as a class.
- Use a number chart so children can see what number symbols look like visually.

For more activities related to counting, go to the K-5 Counting Center.

Suggested Activity:

To practice counting on, begin with a number and context the child recognizes, like four bear manipulatives. Add another bear and say, “five,” then add another bear and say, “six.” This helps students connect the number names with the increase in objects.

Help students see the patterns and make connections. Ask probing questions such as, “How do you know?” and “How did you figure that out?”



One-to-one correspondence is a developmental skill for young children that we take for granted. When a child touches each toy and says the number name out loud, “one, two, three…,” he/she demonstrates the ability to count with one-to-one correspondence. To develop this skill in young children, you can point to objects as you say the number word, or move each object as you say the number word out loud.

Suggested Activity:

To develop one-to-one correspondence, ask students to match two different types of objects together, such as counting bears and a die that shows five dots.

To be successful in mathematics, students must understand **cardinality** by counting various objects and hold that number "in their head." A child who understands the cardinality concept will count a set once and not need to count it again.

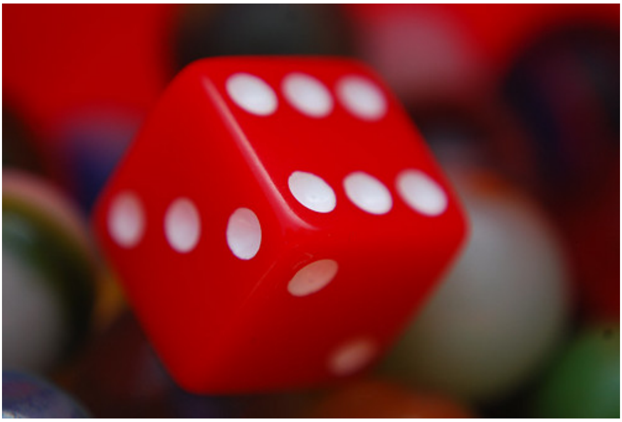
To develop this skill, students need constant repetition of counting and teaching through modeling. Children learn how to count (matching counting words with objects) before they understand that the last word stated in a count indicates the amount of the set. Counting should be reinforced throughout the day, not just during "math time."

Suggested Activity:

Show students a collection of counters. Ask, "How many are here?"

Other examples:

- Ask students to count the number of empty chairs in the classroom.
- Count the number of groups.
- Count the number of crayons in their desk.



Next in the progression of counting is **subitizing**, the ability to quickly identify the number of items in a group without counting the items. Developing the ability to subitize is critical to the success of early number concepts. Subitizing allows children to see groupings, and these groupings are foundational to addition, subtraction, multiplication and division.

Christina Tondevold (2019) explains: *"It's the same thing when kids are learning to read. When we first teach kids to learn to read, they do it in isolation. C-A-T. And then they put that together to make cat. It's the same thing as 1, 2, 3, that's 3. But eventually, we don't want them saying "C-A-T," and having to sound it all out the whole way. Like for catnap, you don't want them to sound out "c-a-t-n-a-p." You want them to see the chunks. Cat-nap."*

In mathematics, when adding 8 + 7, students may not immediately know the sum. But instead of counting on fingers, students can visualize these amounts. Students may see the 7 as 5 and 2 and the 8 as 5 and 3. They can then put together the 5 and 5 and the 2 and 3 to know 10 and 5 is 15.

Games are an excellent way to develop subitizing skills in young students; dice, dominoes, playing cards, etc. Watch the subitizing video, as groups of objects quickly flash on the screen.

Suggested Activity: Subitizing with Paper Plates

Hold up a paper plate with dots on it for 3 seconds and say, "How many dots did you see? What does the pattern look like?" Spend time discussing the configuration of the pattern and how many dots are on the plate. Then show the plate again so children can self-check. (Van de Walle, Karp, Bay-Williams, 2019)

The goal is for the children to recognize the shape of the dots on the plate and when held up, they will recognize that it's a five or a 9 relatively quickly. You want the children to get past one to one counting of the dots and to recognize the number by the dot arrangement. Think of how you recognize the number on dice, you don't count the pips but you know when you see a 5. This is what you want your children to learn.

NUMBER SENSE

Number sense is defined as a "good intuition about numbers and their relations. It develops gradually as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms" (Howden, 1989). When students consider the context of a problem, look at the numbers in a problem, and make a decision about which strategy would be most efficient in each particular problem, they have number sense. Number sense is the ability to think flexibly between a variety of strategies in context.

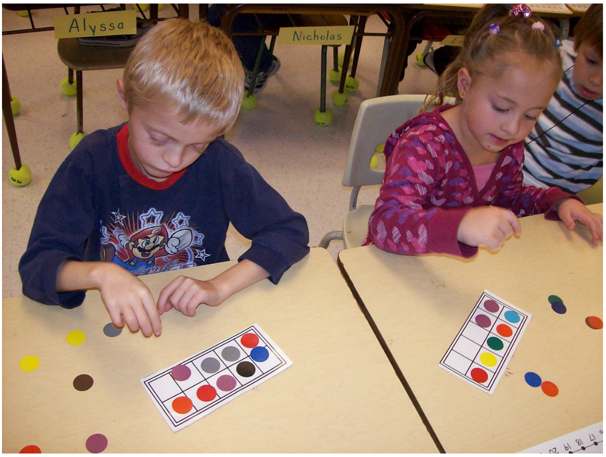
Building on subitizing, composing and decomposing numbers, using ten-frames, and the hundreds chart will help develop number sense in your students. When students have number sense, they gain **computational fluency**, are **flexible** in their thinking, and are able to choose the most efficient strategy to solve a problem.

Number sense is a foundational idea in mathematics; it encourages students to think flexibly and promotes confidence with numbers. According to Marilyn Burns (2007), "students come to understand that numbers are meaningful and outcomes are sensible and expected." And "just as our understanding of phonemic awareness has revolutionized the teaching of beginning reading, the influence of number sense on early math development and more complex mathematical thinking carries implications for instruction" (Gersten & Chard, 1999).

Number sense is a skill that allows students to work with numbers fluently and efficiently. This includes understanding skills such as quantities, concepts of more and less, symbols represent quantities (8 means the same thing as eight), and making number comparisons (12 is greater than 9, and three is half of six). Students must

1. Understand numbers, ways of representing numbers, relationships among numbers, and the number system,
2. Understand the meanings of operations and how they related to one another, and
3. Compute fluently and make reasonable estimates (NCTM, 2000).

Teachers must give students many opportunities to develop number sense through activities with physical objects, such as counters, blocks, or small toys. Most children need that concrete experience of physically manipulating groups of objects. After these essential experiences, a teacher can move to more abstract materials such as dot cards.

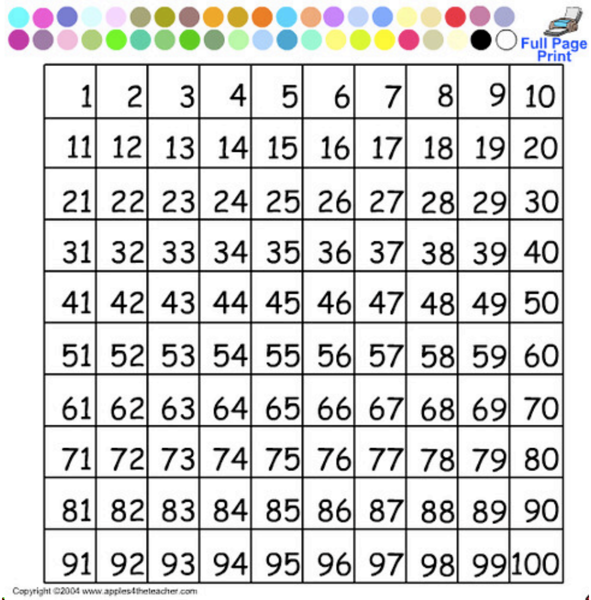


Ten-Frames

Another way to develop number sense in children is to use ten-frames. A "ten-frame" is a rectangle, separated into two rows, with 10 equal spaces. Here is a blank ten-frame from NRICH.

Ten-frames help students develop number sense, a sense of ten, and learn basic number facts. A ten-frame also gives students a visual of numbers, of composing and decomposing numbers, and is a good way to teach addition and subtraction within 10.

This Ten Frame applet from NCTM Illuminations is a fun way for students to practice counting and addition skills.



1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Hundreds Chart

A Hundreds Chart is essential in every K-2 classroom. Ask students what patterns they see on the Hundreds Chart. You can also begin to “hide” numbers on the Hundreds Chart and ask students to tell what that number is and how they know.

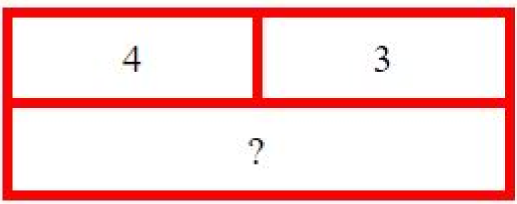
Part	Part
Whole	

Part-Part-Whole Relationships

Teaching part-part-whole relationships is critical to understanding addition and subtraction concepts. The part-part-whole model shows that two parts make up a whole. The following examples show the value of part-part-whole relationships.

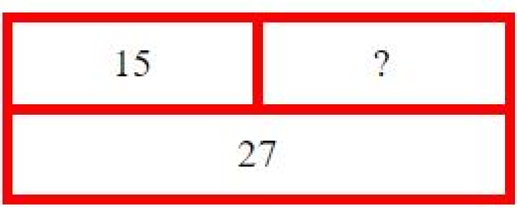
Example #1: *Samantha has 4 red marbles and 3 blue stickers. How many stickers does Samantha have?*

To use the part-part-whole model, think first about the unknown. Since we want to know how many stickers Samantha has in all, that is the unknown or the "whole."



Example #2: *Addison has 27 chocolate candies. 15 are red, and the rest are green. How many green candies does Addison have?*

To use the part-part-whole model, think first about the unknown. Since we want to know how many of Addison's candies are green, that is the unknown or the "part."

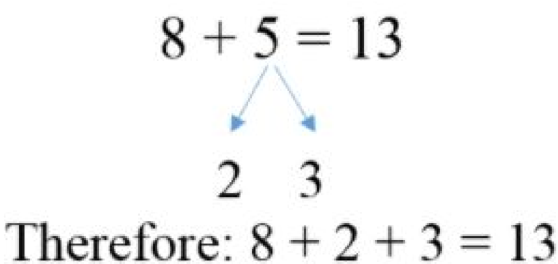


Basic facts are any number or mathematical fact or idea that is instantly recognizable or retrievable without having to resort to a strategy or a calculation. Just knowing basic facts is not enough. Teachers must help students develop the ability to quickly and accurately understand the relationships between numbers.

Students need to make sense of numbers as they find and make strategies for joining and separating quantities.

Students use properties of addition (the **commutative** and **associative** properties) to add whole numbers. They can then use other strategies such as making tens and near doubles. (Student do not need to know the names of the properties, but they should understand them.)

In the strategy "making tens," students decompose numbers to find a 10. For example, when learning 8 + 5, students may decompose the 5 into 2 and 3, and then see that 8 + 2 is 10; therefore 10 + 3 = 13.

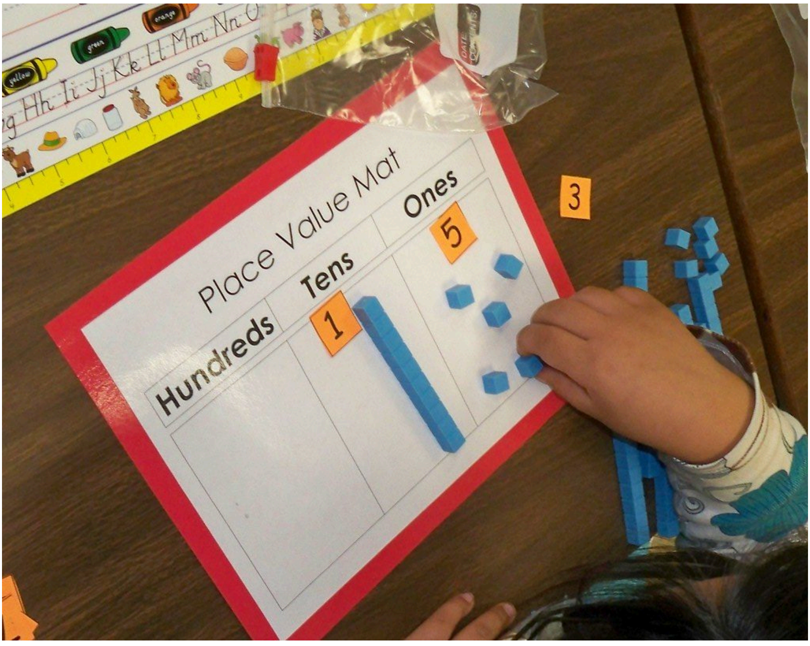


Giving children multiple opportunities to experience early number concepts and number sense is critical for early childhood teachers. Children must experience mathematics through play and manipulatives as they make sense of numbers in their own minds.

CHAPTER 7

Whole Number Place Value

JANET STRAMEL



The focus of chapter 6 was on number sense and a relational understanding of numbers. Number sense is linked to **place**

value and an understanding of the base-ten number system. The progression of place value across grades K-5 is critical for understanding:

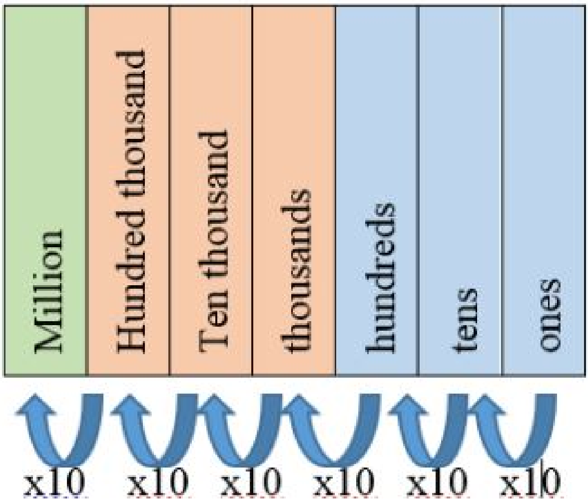
1. Decomposing numbers in base ten
2. Reading and writing numbers
3. Rounding numbers
4. Comparing numbers and quantities in base ten.

By the end of kindergarten, students are expected to count to 100 and count sets of 20 (KSDE, 2017a). In kindergarten, counting is based on a ones approach – the number 15 means 15 ones. According to Wright, et. al. (2006), there is a progression to understanding ten:

1. children understand ten as ten ones,
2. children see ten as a unit, and
3. children easily work with units of ten.

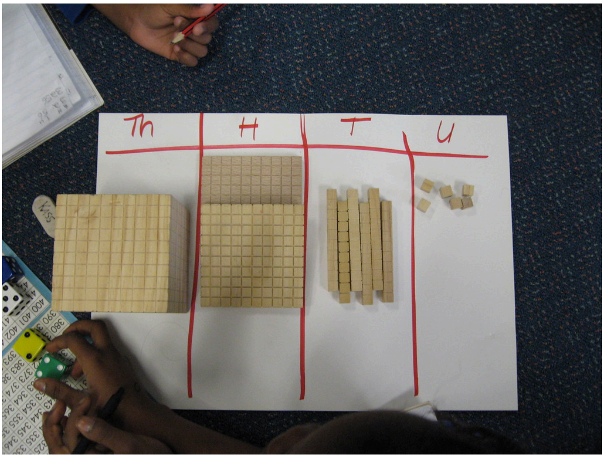
Consider the phrase, “ten ones make one ten.” And now think like a child. How does this make sense? Students in phase one count a set of items and think of that set as ones. When students move to phase two, they begin to see a group of ten as a unit, such as a group of ten ones.

We use the base-ten system; using ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Every number can be represented using these digits. In the base-ten system, the value of each place is always 10 times the value of the place to the immediate right. When moving one place left, the value of the place is multiplied by 10.



In the base-ten system, the “places” are ones, tens, hundreds, thousands, etc. And the digit in each place represents 0-9 of those units. Students learn that ten ones makes a unit, called a ten. They then learn that two-digit numbers are composed of ones and tens.

Place value is a fundamental concept in the elementary grades, and understanding place value is essential in learning mathematics. Place value is the value of the digit in its position. For example, the number 358 has three columns or “places,” each with a specific value. In 358, the 3 is in the “hundreds” place, the 5 is in the “tens” place, and the 8 is in the “ones” place. But more importantly, the value of each digit is 3 hundreds (or 300), 5 tens (or 50), and 8 ones.



Base-Ten Blocks

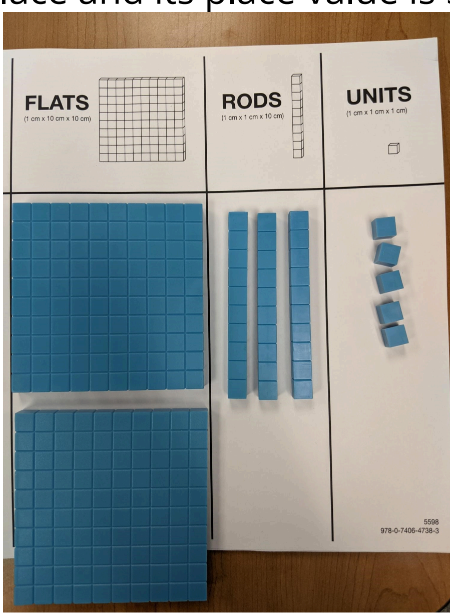
Base-ten blocks provide a hands-on model of our base-ten number system. The smallest cubes are called units. The long, narrow blocks are called rods. The flat, square blocks are called flats. The large cubes are called cubes. The size relationships of the base-ten blocks are perfectly designed to help children discover that it takes 10 units to make one rod, or 10 ones to make one ten; and 10 tens to make one hundred.

Base-ten blocks are ideal for students to physically manipulate "the numbers" so they can conceptually understand the concepts of place value. Nothing can replace the physical **manipulatives** used to help students make the connections from concrete to abstract understanding.

An example showing the relationship among the manipulatives, the place or position, and the place value of the digits in a number is below.

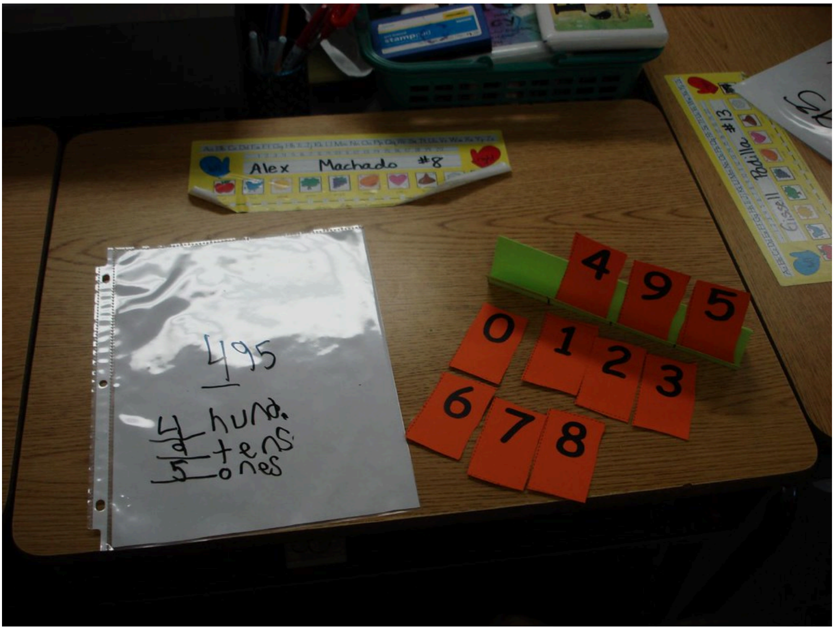
Million	Hundred thousand	Ten thousand	thousands	hundreds	tens	ones
				2	3	5

The number 235 is written in **standard notation**. This shows the "value" of each digit. There are two hundreds, three tens, and five ones. Additionally, in the number 235: 2 is in hundreds place and its place value is 200, 3 is in tens place and its place value is 30, 5 is in ones place and its place value is 5.



Understanding the place value of digits in

numbers helps in writing numbers in their expanded form. For instance, the **expanded notation** of the number 235 is 200 + 30 + 5.
Read more about the Base-Ten Number System by clicking here.



Developing Whole Number Place Value Concepts across the Grades

When thinking about place value and base-ten understanding, first consider the 5 Strands of Mathematical Proficiency from the National Research Council (2001).

Mathematical proficiency is based upon five interwoven components:

- Conceptual understanding – comprehension of mathematical concepts, operations, and relations

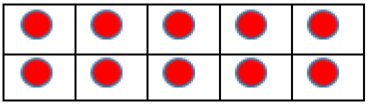
- Procedural fluency – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- Strategic competence – ability to formulate, represent, and solve mathematical problems
- Adaptive reasoning – capacity for logical thought, reflection, explanation, and justification
- Productive disposition – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (NRC, 2001, p. 116).

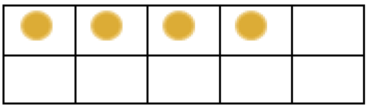
Read more about the 5 Strands of Mathematical Proficiency by downloading the book, "Helping Children Learn Mathematics."
According to James Brickwedde (2012), "place value and the base ten system is an early and easy entry point for students to begin to explore this agility. Without this level of flexibility and fluency, students are limited to inefficient strategies or are overly dependent upon tactical procedures they know only through rote application."

Your language matters when teaching students about place value. Speak in value, not in digits. For example, the value of the 2 in 26 is two tens or 20.

Kindergarten

Kindergarten is the first time many students work with numbers greater than 10 using manipulatives and/or drawings. Kindergarten students separate (decompose) a set of 11-19 objects into a group of 10 and some other ones. Experiences with double ten frames will help students to understand this concept.





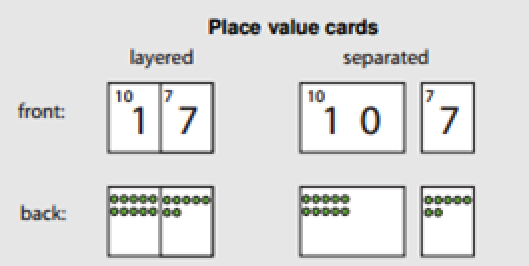
Notice the grouping of the ten. It is obvious that there is one group of ten with some others left over. Students will immediately see that 14 is made up of one ten and four ones.

The "teen" numbers are one group of ten and some more ones. Notice the "teen" numbers that do not follow a particular pattern in the counting sequence. First, think of the number names 11-19: eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen.

- Eleven and twelve are special number words that do not have "teen" as a suffix.
- The verbal counting of "teen" numbers is backwards: the ones digit is said before the tens digit. For example, 37 is read as thirty-seven; tens to ones. But the number 14 is read as fourteen; read ones to tens.

When teaching the "teen" numbers, ask students to read the number as well as describe the quantity. For the number 18, students should read "eighteen" and then say, "18 is one group of ten and eight ones." Additionally, students should record the number sentence 18 = 10 + 8.

By the end of kindergarten, students should be able to compose and decompose numbers between 11 and 19 into tens and some ones. When teaching to compose and decompose numbers, students must use manipulatives and drawings.



Source: KSDE Flip Book Kindergarten (2017b)

Check out the task "What Makes a Teen Number?" at the Illustrative Mathematics website. This activity helps students decompose teen numbers using ten-frames and number sentences.

First Grade

In the first grade Domain Number and Operations in Base Ten, the second Cluster is understand place value. Thinking about *"10 ones makes 1 ten"* is confusing for students. In first grade, students should see 10 ones as 1 ten and that 1 ten has the same value as 10 ones.

Students need many opportunities to practice grouping 10 ones into a bundle of 10. In first grade, students begin to **unitize**. They see that a group of ten objects is also one ten.

Watch this video "One Is One, Or Is It?" narrated by Christopher Danielson for more information on unitizing.

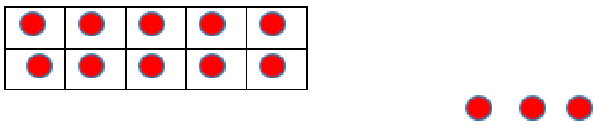
Students must be given numerous experiences using ten-frames, snap cubes, or other groupable models to help develop the concept of place value.

Suggested Activity using Ten-Frames:

Give students 13 counters.

Teacher: “Do you have enough to make a ten?” “Would you have any left over?” “If so, how many left overs do you have?”

Student: “I have filled up one ten-frame and have 3 counters left over. The number 13 has 1 ten and 3 ones.



In first grade, students explore the decade numbers (10, 20, 30, 40, 50, 60, 70, 80, 90) as groups of ten with no ones left over. Students use manipulatives to group, or bundle, groups of ten. As students count any number up to 99, they should group/bundle tens and some more. Furthermore, they should focus on the mathematical language associated with the quantity. For example, the number 62 can be expressed as 62 ones; or 6 groups of ten with 2 left over.

In addition, students should read numbers in standard form AND using place value concepts. Read the number 62 as “sixty-two” as well as 6 tens and 2 ones.

First grade students “compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the relational symbols >, <, =, and ≠”. According to Dougherty, Flores, Louis, & Sophian (2010), students should determine whether quantities are equal or not equal before using “greater than” and “less than.” Once students know the two quantities are not equal, then teachers can have a discussion about “how” they are not equal by asking the following questions. “Which one is greater?” “Which one is less?” These discussions must occur before using the greater than and less than symbols.

Give students extensive experiences to explain their thinking using words and models before using the symbols. Students can use manipulatives to model the two numbers, as well as pictures or number lines. Again, it is critical that students develop the concept of composing and decomposing tens and ones. This is foundational to understanding place value and involves number relationships and promotes a flexibility with mental computation. Furthermore, students must learn through a progression of representations. They begin with concrete models, then move to pictorial models, and then abstract models.

Second Grade

In second grade, students work on decomposing numbers by using place value. Give students extensive experiences with manipulatives as well as pictorial representations of numbers. Students also should be able to decompose numbers into hundred, tens, and ones in several different combinations. For example, 268 could be shown as 2 hundreds, 6 tens, and 8 ones. But is also correct to show 268 as 26 tens and 8 ones, OR 1 hundred, 16 tens, and 8 ones. There are several ways to represent the number 268:

Building on their work in first grade, second grade students need multiple opportunities to count and bundle groups of hundreds. It is critical that students understand that 100 is 10 tens as well as 100 ones.

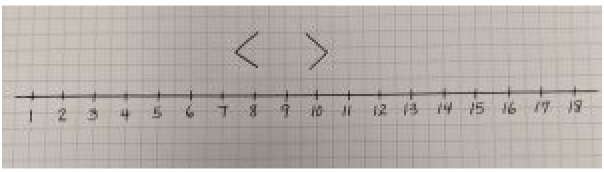
Students in second grade also learn to read, write, and represent a number of objects in various forms

- Base-ten numerals – 268
- Number names – two hundred sixty-eight
- Expanded notation – 200 + 60 + 8
- Unit form – 2 hundreds, 6 tens, 8 ones

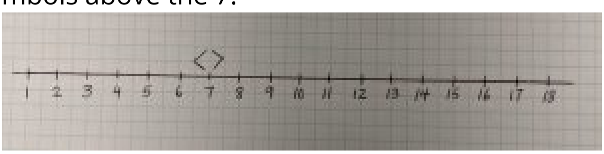
Notice, in the number names, the word “and” is not used between any of the whole number words. Save the word “and” for decimals.

Second grade students build on their work in first grade to compare two numbers and use the relational symbols of <, >, =, and ≠. Give students extensive experiences to explain their thinking using words and models before using the symbols. Students can use manipulatives to model the two numbers, as well as pictures or number lines.

Another way to ensure that students develop a conceptual understanding of the relational symbols (<, >, =, and ≠), use a number line. Be sure to put the greater than and less than symbols above the number line.



Then share the two numbers you want to compare; for example 14 and 7. Explain to students that you want to compare where 14 is **in relation to** the number 7. So you can move the two relational symbols above the 7.



Ask students where 14 is **in relation to** 7. Is it to the left, which would be lesser number? Or is it to the right, which would be greater number?

Using **relationships** truly focuses on the numbers and on tricks.

Additionally, second grade students fluently add using strategies based on place value. For example, using the place value strategy for 56 + 27, a student might say:

- I decomposed 56 and 27 into tens and ones.
- 5 tens plus 2 tens is 7 tens.
- Then I added the ones; 6 ones and 7 ones is 13 ones.
- Then I combined the tens and ones. 7 tens plus 13 ones is 83.

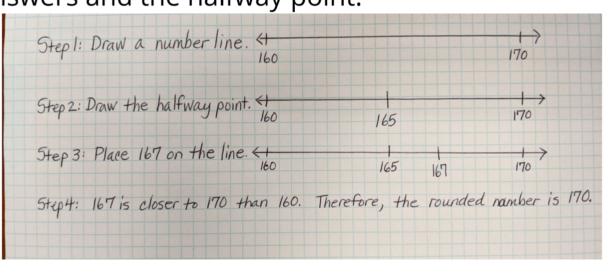
Source: 2017 Kansas Mathematics Standards Flip Book 2nd Grade

Rounding Whole Numbers

The first time students encounter rounding whole numbers is in third grade. Students "use place value understanding to round whole numbers to the nearest 10 or 100." Additionally, students should have a deep understanding of place value and number sense in order to explain and reason about their rounded answers.

When teaching students about **rounding**, you are deciding which number is closest. For example, 127 is closer to 100 than it is 200. If students don't know this, then there is a gap in their place value understanding.

A strong understanding of place value is critical for students to understand rounding of whole numbers. Don't begin by telling students the rules; use a number line so that students can develop a conceptual understanding of rounding. Students need to learn when and why to round numbers by first identifying possible answers and the halfway point.



Using an **open number line** to teach students about rounding will help both their rounding skills and strengthen place value understanding. For example, *round 167 to the nearest ten.*

Source: KSDE Flip Book 3rd Grade (2017c)

Teaching Greater Than and Less Than

Don't teach greater than and less than using methods such as "Pac Man" or "Alligator" or other aids. When you use these, students don't grasp the full meaning of the relational symbol. If you must use one of these "learning aids," be sure to include the mathematical name and symbol with it. But first, explicitly teach that the symbols have names. Second, ask students to read the entire inequality, reading the numbers and symbols left to right, in the same way they would read a sentence.

For example, when students write $5 < 8$, they would read this sentence as "five is less than eight." This is a critical step in learning these relational symbols. Additionally, when students read the inequality aloud, they can recognize errors. For example, $5 < 8$, if students say five is greater than eight, that does not make sense and they can correct their mistake.

Calculators in the Elementary Mathematics Classroom

The NCTM (2015) Position Statement, "Calculator Use in Elementary Grades," states:

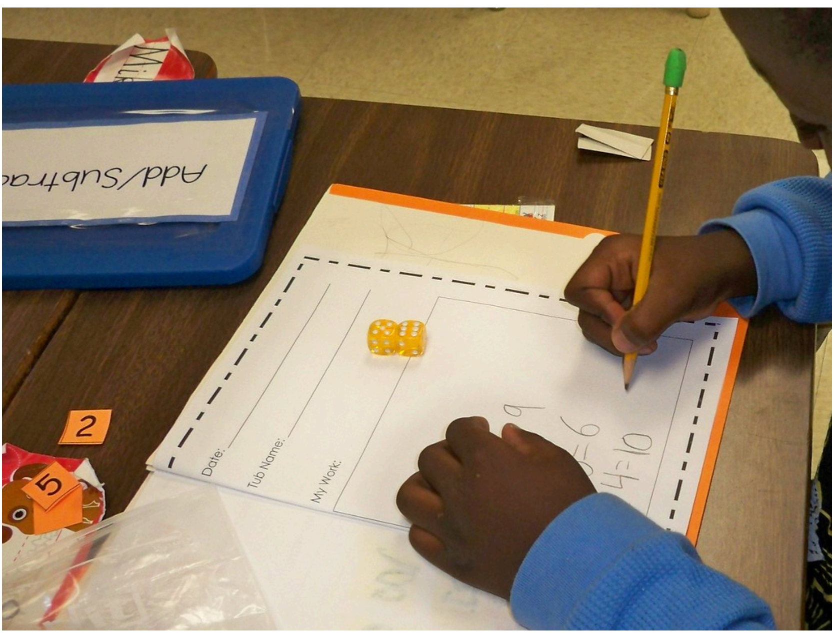
Calculators in the elementary grades serve as aids in advancing *student understanding without replacing the need for other calculation methods. Calculator use can promote the higher-order thinking and reasoning needed for problem solving in our information- and technology-based society. Their use can also assist teachers and students in increasing student understanding of and fluency with arithmetic operations, algorithms, and numerical relationships and enhancing student motivation. Strategic calculator use can aid students in recognizing and extending numeric, algebraic, and geometric patterns and relationships.*

Calculators in the elementary classroom are essential in helping students make sense of mathematics and reason mathematically. Teachers need to plan for the strategic use of calculators that will support student thinking and assist them in making connections to real-world situations.

Read NCTM's Position Statement "Calculator Use in Elementary Grades."

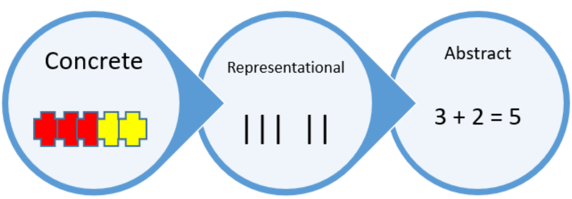
CHAPTER 8

Whole Number Computation



Computational fluency is the ability to accurately, efficiently, and flexibly compute with basic operations (National Research Council, 2001) and is developed over time. In elementary school, students learn about the four basic operations, the properties of the operations, and the strategies to perform those operations. It is developed progressing from concrete models to drawings and then to symbols. The Concrete Representational Abstract (CRA) approach helps students make the connection from concrete manipulatives to abstract mathematical ideas. The three stages of learning mathematics are:

- Concrete – students physically manipulate objects to solve a mathematics problem
- Representational (or semi-concrete) – involves using images to represent objects.
- Abstract (or symbolic) – using only numbers and symbols to solve math problems.



CRA is grounded in the constructivist theory of learning. Teachers begin with concrete manipulatives, transition to visual representations/drawings, and then move to abstract mathematical numbers and symbols. When teaching students addition and subtraction, begin with models/manipulatives, and then progress to bar diagrams, and number lines.

Watch the video from Graham Fletcher, "The Progression of Addition and Subtraction."

Addition and Subtraction Computation

In kindergarten, students developed an understanding of addition as putting together and adding to, and subtraction as taking apart and taking from. Students in first grade are expected to add two-digit numbers to one-digit numbers; and in second grade, students add two- and three-digit numbers. It is important to note that the traditional algorithms are not required until fourth grade. Therefore, students are expected to compose and decompose numbers and use flexible methods to solve problems.

When adding two or more numbers, the result is the **sum**. The **symbol used for addition** is the plus sign (+). The inverse of addition is **subtraction**, and the result is called the **difference**. The symbol used for subtraction is the minus sign (-).

First grade students will extend their number facts and place value strategies to add within 100. Do not introduce the traditional addition or subtraction algorithms in first grade. Instead, represent a problem using words, numbers, pictures, manipulatives, and/or symbols.

Using Base Ten Blocks to Add and Subtract

Base Ten Blocks provide a concrete model of our **base ten number system**. Base-ten blocks provide a hands-on model of our base-ten number system. The smallest cubes are called units. The long, narrow blocks are called rods. The flat, square blocks are called flats. The large cubes are called cubes. The size relationships of the base-ten blocks are perfectly designed to help children discover that it takes 10 units to make one rod, or 10 ones to make one ten; 10 tens to make one hundred; and 10 hundreds to make one thousand.

Base-ten blocks are ideal for students to physically manipulate "the numbers" so they can conceptually understand the concepts of place value.

Examples:

Addition Example: 53 + 38 =

- Students begin by placing 53 base-ten blocks (5 rods and 3 units) and 38 base-ten blocks (3 rods and 8 units) on their table.

- Students will then combine groups of ten ones and trade that group for 1 ten/rod.

Subtraction Example: 135 – 48 =

- Students begin by placing 135 base-ten blocks on their table.
- Students will then remove 48 from this set of base-ten blocks. Students will notice immediately that they can not remove eight ones/units because there are only 5 on the table. Therefore, they will need to regroup/exchange/trade one ten/rod for ten ones/units so they can take away 8 ones.
- Then regroup/exchange/trade the one hundred/flat for ten tens/rods.

Addition and Subtraction Based on the Properties of Operations

In first grade, students applied the properties of operations to add and subtract. They can then build on these understandings to fluently add and subtract using the properties, such as the Commutative Property of Addition and the Associative Property of Addition.

Addition Example: 37 + 25

- *I decomposed the 37 and 25 into tens and ones so now I have 30 + 7 + 20 + 5.*
- *I add 30 and 20 first to get 50.*
- *Then I add 7 and 5 to get 12.*
- *Then 50 + 12 is 62.*

Addition and Subtraction with Bar Diagrams

Another model to use when adding and subtraction is the bar diagram. Bar diagrams (sometimes called strip or tape diagrams) are a pictorial representation of a problem shown by bars or boxes. Using bar diagrams is the representational/semi-concrete link between the concrete and the abstract phases of learning..

Note the following examples:

Susan made 24 cookies for the school carnival. Terry made 36 cookies. How many cookies will the school carnival have to share?

24	36
?	

Kathryn has read 17 books. Her goal is to read 52 this year. How many more books does she need to read?

17	?
52	

Addition and Subtraction with Open Number Lines

An open number line has no numbers or marks on it; it is used when students want to share their thinking strategies. Before using the open number line, students must have a strong understanding of numbers up to 100. Prerequisite skills include counting on and counting back, knowing basic addition and subtraction facts to ten, and adding/subtracting a multiple of 10 to or from any two-digit number.

To introduce the open number line, ask students questions such as:

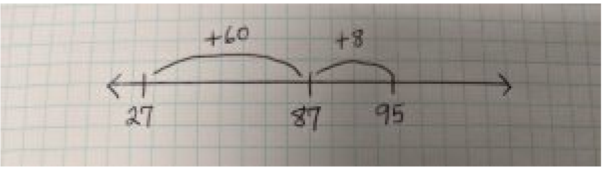
- How can you go from 0 to 37 in the least number of jumps of tens and ones?

Encourage students to share their strategies and discuss which strategy is most efficient. For example, when jumping from 0 to 37, one student could say, “three jumps of ten and seven jumps of one” and another student could say, “four jumps of ten and back three ones.”

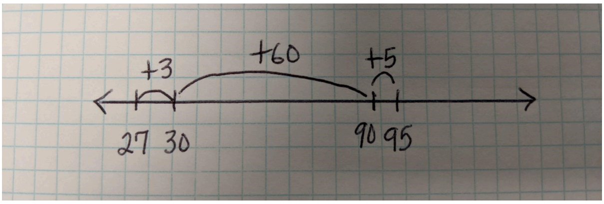
When students have mastered questions like the one above, the next phase in the use of the open number line is with addition and subtraction. Be sure to give students the flexibility to choose what types of jumps they want to use. Some possible questions are:

- How can you go from 26 to 52 in a small number of jumps? Who has another strategy?
- How can you go from 61 to 37 in a small number of jumps? Who has another strategy?
- How can you solve 27 + 85?
- How can you solve 68 – 31?

To solve the problem 27 + 68, one student may begin at 27, add 60 and then add 8.

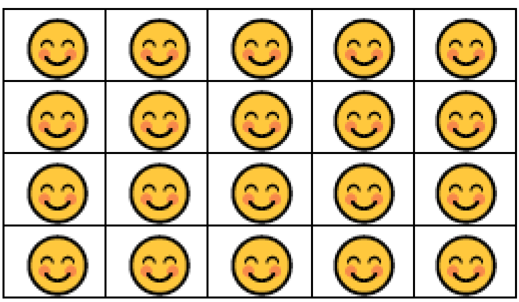


Another student may solve 27 + 68 by adding 3, then add 60, then add 5.



Multiplication and Division Computation

Second grade students will "use addition to find the total number of objects arranged in rectangular arrays" (KSDE, 2017). Rectangular arrays are an arrangement of objects in horizontal rows and vertical columns and provides a visual model for multiplication. One way to introduce students to arrays is to ask them to build all possible arrays with no more than 25 objects, and up to 5 rows and 5 columns (5 x 5 array). In addition, ask students to draw their arrays on grid paper and write two different equations under their array. An example of this is a 5 x 4 array:

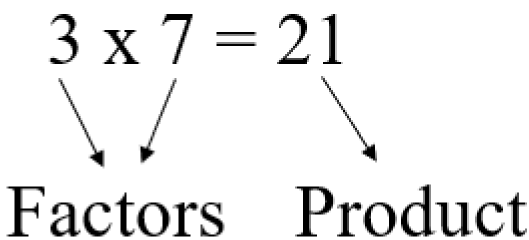


- The equation by rows is 20 = 5 + 5 + 5 + 5
- The equation by columns is 20 = 4 + 4 + 4 + 4 + 4

Source: 2017 Kansas Mathematics Standards Flip Book 2nd Grade

Students must explore the concept of rectangular arrays using concrete objects, as well as pictorial representations on grid paper. In addition, students will use the **commutative property of addition** since they can add either the rows or columns and get the same answer.

In third grade, students will find the products of whole numbers and recognize multiplication as finding the total number of objects when there are an equal number of groups. Refer to "groups of" objects, such as 4 x 7 is 4 groups of 7. Begin using the vocabulary terms **factor** and **product** at this level.



Be sure to refer to Table 2 in the Kansas Mathematics Standards for the types of multiplication problems.

Students must experience all three types of multiplication problems: size of group is unknown, unknown product, and number of groups is unknown. When students are solving problems, ask them to also write the related multiplication or division equation. For example, to determine the unknown number in the equation 24 ? = 3, students should use their knowledge of the related multiplication fact 3 x 8 = 24.

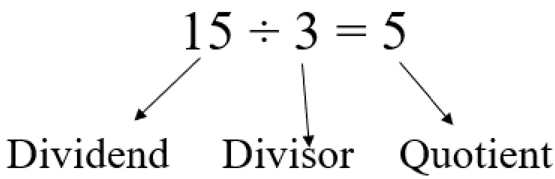
Watch this video from Graham Fletcher, "The Progression of Multiplication"

When teaching division, focus on two models of division: partitive and measurement.

- **Partitive division** focuses on the question, "How many are in each group?" An example of the partition model in context is, "There are 15 cookies on the counter. If you are sharing the cookies equally among 5 friends, how many cookies will each friend receive?"
- **Measurement division** (sometimes called repeated subtraction) focuses on the question, "How many groups can you make?" An example of the measurement model is, "There are 15 cookies on the counter. If I give each

person 3 cookies, how many people can I give cookies to?"

In third grade, students should be introduced to the vocabulary terms for division: **quotient**, **dividend**, and **divisor**. In addition, students should use models and concrete objects to justify their thinking.



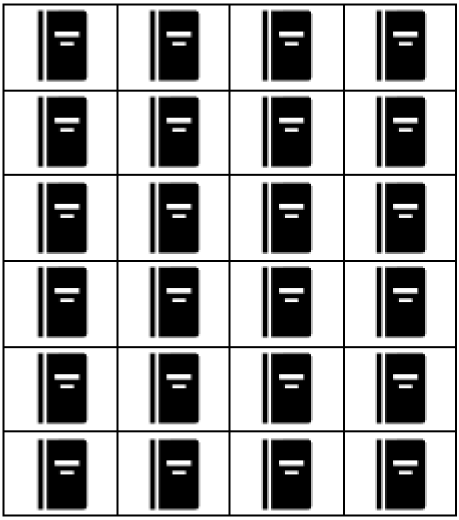
In third grade, students use various strategies to solve word problems. Expect students to use a variety of representations when solving problems, such as rectangular arrays, drawing pictures of equal groups, mental math, number lines, and equations.

Watch this video from Graham Fletcher, "The Progression of Division."

Examples of Multiple Strategies for Teaching Multiplication in Third Grade:

There are 24 books to be displayed on the library shelves. If the books are displayed with 4 books on each shelf, how many shelves would be needed?

Rectangular Array

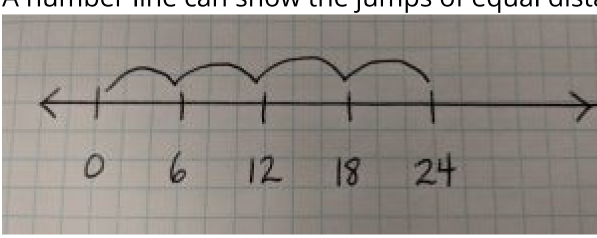


Mental Mathematics

A student could reason that "6 and 6 are 12" and "12 and 12 are 24." Therefore, there are 4 groups of 6 for a total of 24 books.

Number Line

A number line can show the jumps of equal distance.



Equations

Students may use related equations, such as

- 4 x 6 = ?

- 6 x 4 = ?
- 24 × ? = 4
- 24 ÷ 6 = ?
- ? ÷ 4 = 24
- ? ÷ 6 = 4

Furthermore, third graders should be introduced to **variables**. Letters for variables should begin being used in third grade for the unknown number, for example, 24 ÷ x = 4.

When students use various strategies and multiple representations, you are helping them develop **multiplicative reasoning**. Multiplicative reasoning is the ability to work flexibly with the concepts, strategies and representations of multiplication (and division). It helps students see the different kinds of relationships between numbers, whereas additive thinking only requires a capacity to work with the numbers themselves (Siemon, 2015). Multiplicative reasoning is related to iterating, or making multiple copies.

Examples of Multiple Strategies for Teaching Division in Third Grade:

The two strategies students can use to solve division word problems are using diagrams and/or pictures (partitive division) or determining the number of shares (measurement division).

This is an example of partitive division since the size of the groups is unknown. *Shelby has 36 candies to share equally with her 4 friends. How many candies will each person receive?*

This is an example of measurement division since the number of groups is unknown. *Amie loves sodas and buys one every afternoon. Each soda costs $2. If she has $12, how many days can she buy a soda?*

Starting	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6
$12	12 – 2 = 10	10 – 2 = 8	8 – 2 = 6	6 – 2 = 4	4 – 2 =2	2 – 2 = 0

Therefore, it took 6 days for Amie to spend her $12 on sodas.

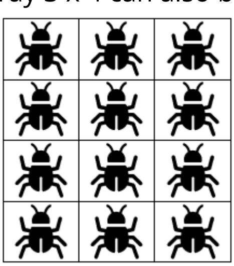
Properties of Multiplication

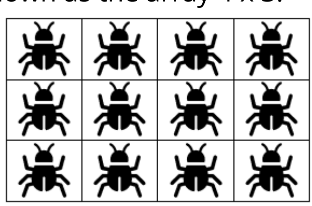
Students in third grade are expected to understand the properties of multiplication, although they do not need to use the formal terms. Teachers must use the correct terms.

The four mathematical properties of multiplication that students need to be familiar with and use:

- **Commutative Property of Multiplication**,
- **Associative Property of Multiplication**,
- **Multiplicative Identity Property**, and
- **Distributive Property of Multiplication over Addition**.

The **Commutative Property of Multiplication** is when two numbers are multiplied, the product is the same regardless of the order of the factors. To “commute” means to travel, and according to the Commutative Property of Multiplication, the numbers can “move around” without changing the product. For example, the array 3 x 4 can also be shown as the array 4 x 3.





The **Associative Property of Multiplication** is when three or more numbers are multiplied, the product is the same regardless of the grouping of the factors. “To associate” means to connect or join with something; and according to the Associative Property of Multiplication, numbers can “associate” with any other number. For example, 6 x 5 x 2 could be solved the following ways:

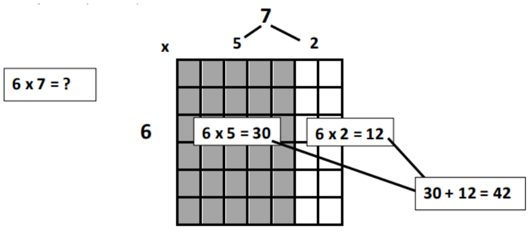
(6 x 5) x 2

- Multiply 6 x 5 first = 30

OR 6 x (5 x 2)

- Multiply 5 x 2 first = 10
- Then 30 x 2 = 60 Then 6 x 10 is 60.

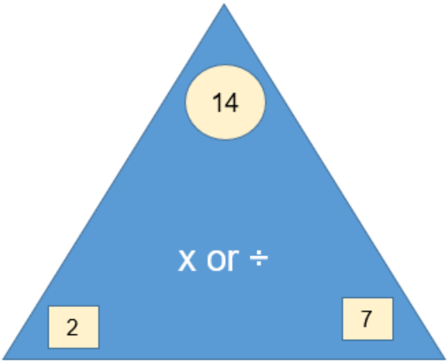
The **Distributive Property of Multiplication over Addition** is when multiplying a sum by a number is the same as multiplying each addend separately and then adding the products. To “distribute” means to spread out equally. For example, 8 x 7 could be shown in a rectangular array and then decompose 7 into 5 and 2.



Source: Kansas Mathematics Standards Flip Book 3rd grade (2017a)

Fact Triangles

Fact Triangles can be used to help students see the inverse relationships between multiplication and division.



2 x 7 = 14 7 x 2 = 14

14 ÷ 2 = 7 14 ÷ 7 = 2

2 = 14 ÷ 7 7 = 14 ÷ 2

Click here for Fact Triangles printables.]

By the end of third grade, students will **fluently** multiply and divide single digit numbers within 100. This does not mean a focus on timed tests, but focus on hands-on experiences, working with manipulatives and drawing pictures. In addition, give students multiple opportunities to experience multiplication and division problems written both horizontally and vertically. Refer to chapter 5 for strategies students can use to develop fluency with basic multiplication and division facts.

Estimation

Don't teach estimation by asking students "make a guess." The word "estimate" comes from the Latin word aestimare, which means to value. To estimate means to judge the value of something, and an estimate is the resulting calculation. A "guess" is to form an opinion with little or no evidence. When you tell students to make a guess, they are making uninformed conclusions. On the other hand, estimating means to make an informed decision based on data and models (Nichol, n.d.).

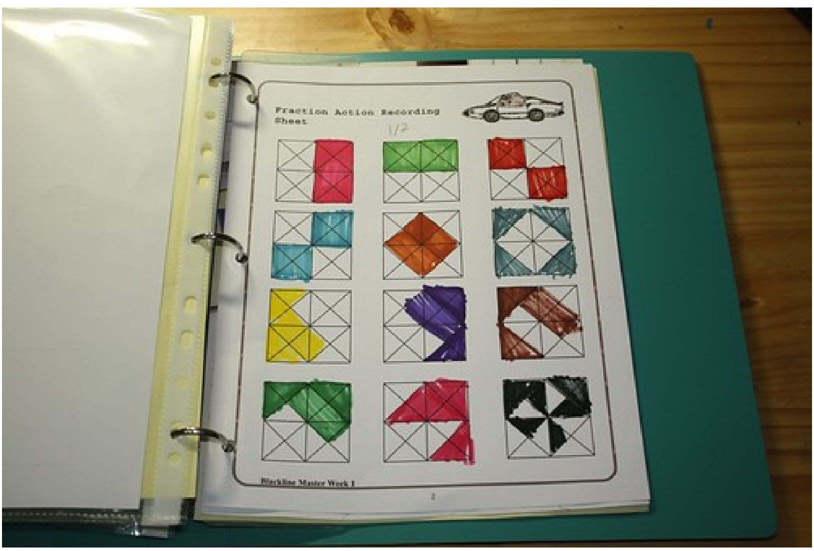
Teaching Traditional Algorithms

Traditional algorithms are digit oriented whereas invented strategies are number oriented. When students invent strategies for solving problems, they are more flexible with the problems. With the traditional algorithm, students see and do the problem following rules. Additionally, with invented strategies, students make fewer errors.

The Kansas Mathematics Standards suggest that students have knowledge of standard/traditional algorithms in fourth grade for addition and subtraction, fifth grade for multiplication, and sixth grade for division. These topics are introduced much earlier than the grade level for the standard algorithm so that students can make sense of why that algorithm works. The key is that students understand the mathematical operations before they learn the "rules" of traditional algorithms.

CHAPTER 9

Early Fraction Concepts



What are Fractions?

According to the Kansas Mathematics Standards (2017), the formal definition of fraction is a number expressible in the form

$$\frac{< em > a < /em >}{< em > b < /em >}$$ a}{*b*}" title="\frac{*a*}{*b*}" class="latex mathjax"> where a is the number of equal parts being referenced and b is the number of equal parts in the whole. But what does this mean?

The word “fraction” comes from the Latin word fractus or broken. A fraction describes how many parts of a certain size there are, for example, one-half, three-fourths, etc. Additionally, the top number (numerator) says how many parts you have, and the bottom number (denominator) says how many equal parts in the whole amount. Most importantly, a fraction is a number.

An understanding of fractions begins in **first grade** when students partition circles and rectangles into two and four equal parts, and describe the parts with the words halves, fourths, and quarters. In **second grade**, students partition circles and rectangles in two, three, and four equal parts, and describe those parts with the words halves, thirds, and fourths. Fractions become the major emphasis in **third grade** as students look at fraction symbols, and explore unit fractions.

“Students need significant time and experiences to develop a deep conceptual understanding” of fractions (Van de Walle, Karp, Bay-Williams, 2019, p. 338). Teachers typically begin fraction instruction with objects, paper folding, or even plastic fraction circles. But often we move too quickly past these models to abstract computation. Some experts even argue that drawings of fractions fail to be concrete enough for some students. The act of cutting and folding and manipulating is critical for all students.

Watch this video from Graham Fletcher, “The Progression of Fractions.”

Fraction Constructs

Fractions include many meanings such as part-whole, measurement, division, operator, and ratio. Understanding fractions includes the need to understand all of these different meanings. As you and your students begin to make sense of fractions, you must also understand all the possible concepts that fractions represent. The following constructs are based on the research from Van de Walle, Karp, and Bay-Williams (2019):

Fractions as Part-whole Comparisons. Part-whole comparisons is a good place to start when building a conceptual understanding of fractions. Part-whole comparisons can be shown by shading a region, part of a group, and measurement.

- **Shading a region**

For example, a circle can be divided into four equal parts. If three of those parts are shaded pink, then the fraction shaded is $\frac{3}{4}$.

- **Part of a group**

For example, I have five pets; three cats and two dogs. The fractional part of cats to all of my pets is $\frac{3}{5}$.

- **Measurement**

For example, in the fraction $\frac{3}{8}$, students use the **unit fraction** $\frac{1}{8}$ to count or measure that it takes 3 of those units to reach $\frac{3}{8}$.

Fractions as Division. Just like with whole numbers, division means making equal-sized groups. Students understand the connection to equal- and fair-sharing which builds from their experiences with sharing and partitioning. For example, "Eight friends are sharing four apples. How much of an apple will each person receive?” Each person will get $\frac{4}{8}$ or $\frac{1}{2}$ of an apple.

A fraction is the quotient of division of two integers. For example, if you want to divide three things equally among 5 people, students know to this can be solved with division, although the quotient of $\frac{3}{5}$ cannot be represented with a whole number, so new numbers need to be introduced; the fraction $\frac{3}{5}$.

Fractions as Operators. This construct builds on the concepts of seeing a fraction as a multiple of a unit fraction. To find $\frac{3}{4}$ of something, you could divide by 4 and then multiply by 3, or multiply by 3 and then divide by 4. The result would be a smaller quantity than the original. For example, $\frac{3}{4}$ x 12 = 9.

Fractions as Ratios. Ratio is another meaning of fractions. For example, $\frac{5}{8}$ can be seen as the probability of an event being five in eight. Ratios can be expressed in two different ways: part-part, such as the ratio of boys to girls in class, and part-whole, such as the ratio of boys to the whole class.

The Language of Fractions

The language of fractions is critical to students developing a conceptual understanding of fractions. But this does not mean that teachers teach only the definitions of fractions, but instead teachers should first focus on making sense of what fractions are. Fractions are a way to describe and represent a part of a whole.

Students first begin to make sense of fractions in first grade when they compose and decompose plane figures, such as putting together two triangles to make a quadrilateral. First grade students must have multiple opportunities and many experiences with the words halves, fourths, and quarters. According to Van de Walle, Karp, and Bay-Williams (2019), the best time to introduce these words to students is during problem solving, not before. For example, when students are cutting a pan of brownies into four equal shares, the teacher can say, “We call these fourths. The whole is cut into four equal-size parts called fourths.”

In second grade, students add to their fraction vocabulary the word “third,” but students are not introduced to the traditional fractional notation until third grade when they develop an understanding of unit fractions. Additionally, the fraction expectations in third grade are limited to fractions with denominators of 2, 3, 4, 6, and 8. It is in third grade that students develop an understanding of fractions as numbers, use the fraction symbols, and use area and linear models. Third grade lays the foundation for an understanding of fractions and is critical for students to be successful in future mathematics.

The **numerator** is the “counting number,” or the number of parts we have.

The **denominator** is how many parts are in the whole.

When talking about these terms, be sure to ask students the following questions:

- “What does the numerator in the fraction tell us?”

- "What does the denominator in the fraction tell us?"
- "What would a fraction equal to one look like?"
- "How do you know if a fraction is less than or greater than one?"

Area – Fractions are determined on how much of a part or area relates to the whole area.

Length – Fractions can be represented on a number line, or as a distance between 0 and another point on the number line.

Set -Fractions can be represented based on how many items are in part of the whole set.

Fractions are Numbers

The most important thing young children need to understand is that fractions are numbers; they represent quantities that have values (Van de Walle, Karp, & Bay-Williams, 2019). **Partitioning** and **iterating** are two important actions that emphasize the numerical nature of fractions.

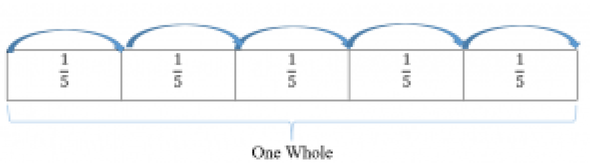
Partitioning is when students section or segment a shape into equal-sized parts. When partitioning, start with a **unit fraction** such as $\frac{1}{3}$. One-third ($\frac{1}{3}$) is the size of piece you would get by taking a whole and dividing it into 3 equal parts.

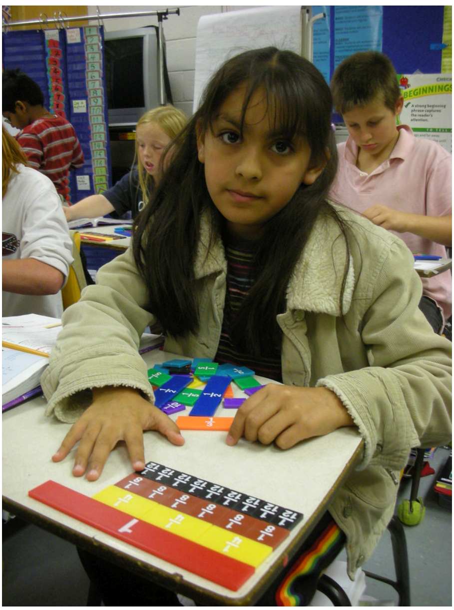
One Whole		
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

When students draw and see diagrams such as the one above, it emphasizes that fractions are a *fraction of something*. You would teach this by saying, *"We call these pieces thirds because we cut the whole into three parts. All of these parts are the same size – thirds."*

Students are introduced to fractions in first grade as they are presented with the names for fractional parts – halves and fourths; in second grade, students partition circles and rectangles into two, three, and four equal parts, and describe those parts using the words *halves, thirds,* and *fourths;* fractional symbols are not introduced until third grade.

Iterating is when students perform the action of aligning, copying, or combining equal units to verify a fraction. According to Siebert (2007), "To conceive of a fraction from an iteration perspective, first start with a unit fraction, such as $\frac{1}{5}$. How can you tell if something is $\frac{1}{5}$? An amount is $\frac{1}{5}$ if five copies of it equals 1. The image here is of taking $\frac{1}{5}$ and iterating is 5 more times to make a whole."



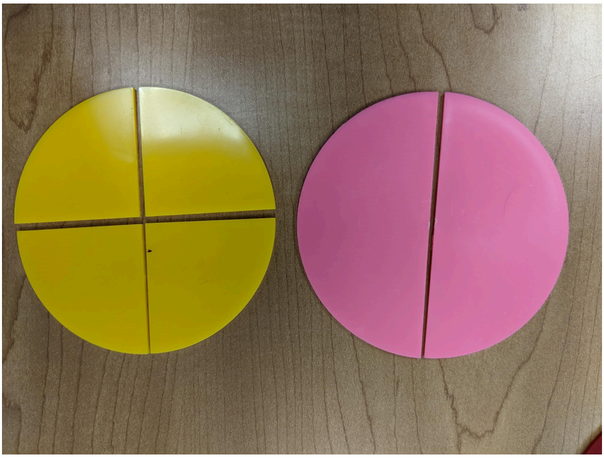


Equivalent Fractions and Fraction Comparisons

Equivalence is a core concept in mathematics. Equivalent fractions have the same value, even though they may look different. For example 12 and 24 are equivalent because they are both "half."

Teachers must devote enough time so that their students have a deep understanding of this concept. Equivalent fractions and comparing fractions is first addressed in third grade, and then applied in fourth grade. The focus of both the third and fourth grade is to use visual fraction models so that students can explore the idea of equivalent fractions instead of using algorithms.

An important concept when **comparing fractions** is both the size of the parts and the number of parts. For example, $\frac{1}{4}$ is lesson than $\frac{1}{2}$ because when the whole is cut into 4 pieces, the pieces are smaller than when the whole is cut into 2 pieces. Students need to see this.



Students are not expected to generate a rule for equivalent fractions until fourth grade, although third grade students should notice the connections between the models and the fractions.

Fraction Equivalence

When teaching fraction equivalence, do not move to the traditional algorithm too quickly. Allow your students to make sense of the relationship between the two fractions first. After students have a conceptual understanding using visual representations and models, then you can ask if they see patterns in the way the fractional parts are counted (Stramel, 2021).

CHAPTER 10

Geometry and Measurement

Geometry is a branch of mathematics that studies the sizes, shapes, positions, and dimensions of things. “Geometric and spatial thinking are important in and of themselves, because they connect mathematics and the physical world... and because they support the development of number and arithmetic concepts and skills” (Progressions for the Common Core State Standards in Mathematics, 2013). The Progressions document goes on to say,

> *“learning geometry cannot progress in the same way as learning number, where the size of the numbers is gradually increased and new kinds of numbers are considered later. In learning about shapes, it is important to vary the examples*

> *in many ways so that students do not learn limited concepts that they must later unlearn. From Kindergarten on, students experience all of the properties of shapes that they will study in Grades K–7, recognizing and working with these properties in increasingly sophisticated ways. The Standards describe particular aspects on which students at that grade level work systematically, deeply, and extensively, building on related experiences in previous years"* (Progressions for the Common Core State Standards in Mathematics, 2013).

Former NCTM President J. Michael Shaughnessy said, "If algebra is the language of mathematics, geometry is the glue that connects it" (Shaughnessy, 2011). Additionally, geometry "covers the skills and concepts of visualization, spatial reasoning and representation, and analyzing characteristics and properties of two- and three-dimensional shapes and their relationships" (Lappan, 1999). In the Kansas Mathematics Standards, geometry spans every grade level from kindergarten to grade eight; it first begins with **spatial sense**, an intuition about shapes and the relationships between them including an ability to recognize, visualize, represent, and transform geometric shapes.

Van Hiele Levels for Teaching Geometry

The development of geometric thinking comes from Pierre van Hiele and Dina van Hiele-Geldof. The van Hieles identified five levels of geometric thinking through which students pass. Most elementary students are at levels 0 or 1 and some middle school students are at level 2. The levels are developmental – children of any age begin at level 0 and progress to the next level through experiences with geometric ideas (Van de Walle, Karp, Bay-Williams, 2019).

Level 0: Visualization

Students begin by recognizing shapes by their whole appearance, but not exact properties. For example, students see a door as a rectangle or a clown's hat as a triangle, but may not be able to recognize the shape if it is rotated. The emphasis at Level 0 is on shapes that students can observe, feel, build/compose, or take apart/decompose. The goal of Level 0 is to explore how shapes are alike and how they are different and use these ideas to create classes of shapes (Van de Walle, Karp, Bay-Williams, 2019).

Level 1: Analysis

At this level, students start to learn and identify parts of figures and can describe a shape's properties. Additionally, students at this level understand that shapes in one group have the same properties. For example, students know that parallelograms have opposite sides that are parallel and can talk about the properties of all parallelograms, not just this one. Students at Level 1 use physical models and drawing of shapes and use the properties of shapes such as symmetry, classification, and congruent sides and angles.

Level 2: Informal Deduction/Abstraction

Students at Level 2 start to recognize the relationship between properties of shapes and develop relationships between these properties. Students will consider if-then reasoning, such as "If all four angles are right angles, the shape must be a rectangle. If it is a square, all angles are right angles. If it is a square, then it must be a rectangle." Level 2 includes informal logical reasoning and should be encouraged to ask "Why?" or "What if?" Additionally, Level 2 tasks emphasize logical reasoning.

Level 3: Formal Deduction

At this level, students analyze informal arguments and are capable of more complex geometric concepts. A student at this level is usually in high school.

Level 4: Rigor

The last level of geometric reasoning is the ability to compare geometric results in different axiomatic systems; they see geometry in the abstract. A student at this level is usually a college mathematics major.

Read this short article from NRICH for some activities required at each of the levels of geometric thinking.

In order to support students as they move from a Level 0 to a Level 1, teachers should focus on the following:

- Focus on the properties of shapes rather than the identification of those shapes,
- Challenge students to test their ideas about shapes using a variety of examples, and
- Provide multiple opportunities for students to draw, build, make, compose, and decompose shapes.

Shapes and their Properties

Students should first focus on the location and position of shapes in order to develop a variety of skills that will contribute to their spatial thinking. In kindergarten, students are expected to describe the position of shapes in the environment using the terms above, below, beside, in front of, behind, and next to. Additionally, students develop spatial sense by connecting geometric shapes to their everyday lives and shapes in their environment.

Students must have experience with a variety of two- and three-dimensional shapes. Additionally, triangles should be shown in several forms and not always with the vertex at the top or the base horizontal with the bottom of the paper (Van de Walle, Karp, & Bay-Williams, 2019).

Kindergarten students learn to describe shapes by their attributes and should be given multiple opportunities to build physical models. When students build and manipulate shapes, they can learn to explore and describe the number of sides and corners (vertices). Ask students to describe shapes by the properties, such as “squares have four sides of equal length.”

Two-dimensional shapes

It is in kindergarten that students learn to distinguish between two- and three-dimensional shapes. Two-dimensional shapes are flat and can be measured in only two ways such as length and width. Examples of two-dimensional shapes are squares, circles, triangles, etc.

Classifying shapes begins in kindergarten. And when students sort and classify polygons, they should determine the groupings, not the teacher. In second grade, students focus on triangles, quadrilaterals, pentagons, and hexagons. Third grade students think about the subcategories of quadrilaterals, and by fifth grade, they “understand the attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category.” For example, a square is a rectangle and a square is also a rhombus.

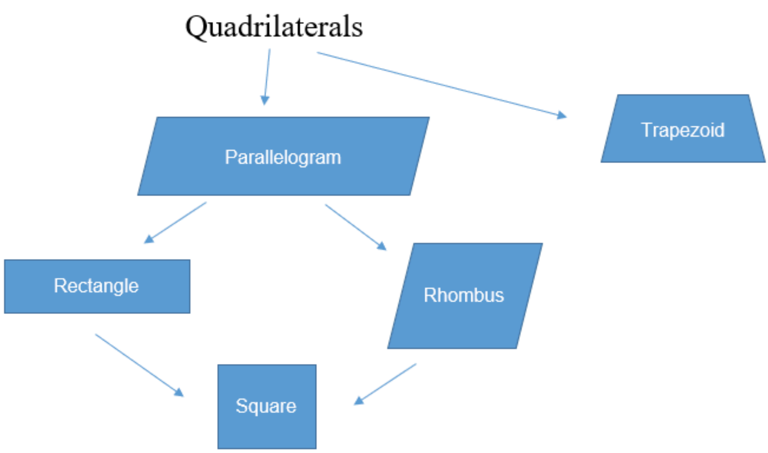
Triangles

Types of triangles are first introduced in fourth grade with the beginning concept of right triangles. A **right triangle** is a triangle with one 90^o angle. Other types of triangles are **acute triangles** and **obtuse triangles.**

Triangles can also be classified by their sides: **equilateral triangle**, **isosceles triangle**, and **scalene triangle**. An equilateral triangle is a triangle in which all sides are the same length. An isosceles triangle has at least two sides of the triangle the same length. And a scalene triangle has no sides that are the same length.

Quadrilaterals

Quadrilaterals are polygons with four sides. The general types of quadrilaterals are **parallelograms**, **rhombi**, **rectangles**, **squares**, and **trapezoids**. There is a hierarchy of quadrilaterals, as shown below.



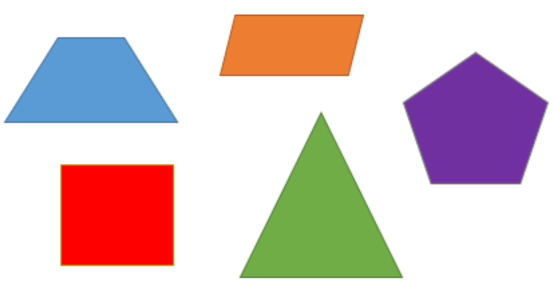
Composing and Decomposing

Students need multiple opportunities to explore how shapes fit together to form larger shapes (compose) and how larger shapes can be made of smaller shapes (decompose). In kindergarten, students move beyond identifying and classifying shapes to creating new shapes using two or more shapes. This concept is foundational to students' development of translation (move or slide), rotation, and reflection (flip).

In first grade, students compose and decompose plane figures and determine which attributes are **defining** and **non-defining.** Defining attributes are those properties that help to define a shape, such as the number of angles, number of sides, length of sides, etc. Non-defining attributes are properties that do not define a shape such as color, location, position, etc. An example is a square. All squares must be closed figures and have 4 equal sides and all angles 90^{o}; these are defining attributes. Squares can be different colors and be turned in a different direction; these are non-defining attributes.

Example:

Which of the following is a square? And how do you know?

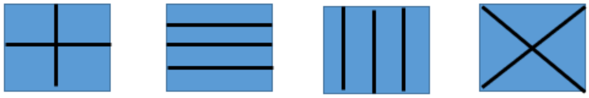


- A square has 4 sides.
- All four sides are the same length.
- All angles are 90^{o}.
- Therefore, the red figure is a square.

Be sure to give students many opportunities with a variety of manipulatives to explore and build shapes. These manipulatives could be paper shapes, pattern blocks, color tiles, tangrams, or geoboards with colored rubber bands (or you can use virtual geoboards).

When students are composing and decomposing both plane and solid figures, they are also building an understanding of part-whole relationships and compose and decompose numbers. Furthermore, it is in first grade that students partition regions. Students need multiple opportunities to use the words halves, fourths, and quarters, which is critical to understanding fractions. In second grade, students work with halves, thirds, fourths; and teachers will help students make the connections that a whole is two halves, three thirds, or four fourths.

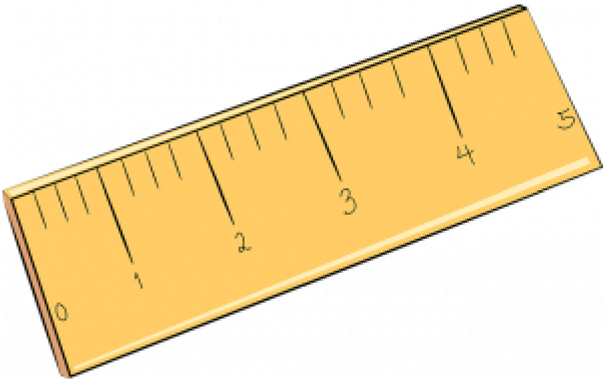
In third grade, students partition shapes into equal portions. For example, partition a square into four equal parts.



Each of the above squares are partitioned into fourths, and each part has the same area.

Three-dimensional shapes

Kindergarten students learned to distinguish between two- and three-dimensional shapes. Three-dimensional shapes have three dimensions, such as height, width, and depth. Examples of three-dimensional shapes are **cubes**, **prisms**, **rectangular prisms**, **pyramids**, etc. Additionally, it is important to note that the faces of three-dimensional shapes can be named as specific two-dimensional shapes. Give students multiple opportunities to work with pictorial representations, concrete objects, and technology as they develop their understanding of both two- and three-dimensional shapes.



Measurement is the "assignment of a numerical value to an attribute of an object" (NCTM, 2000). Measurement is a critical concept in mathematics because of the connection to everyday life. Additionally, there are connections to other mathematics, as well as other content areas.

In kindergarten through grade 2, students compare and order objects using the terms longer and shorter. Length is the focus in this grade band, although students also explore weight, time, area, and volume. In the 3-5 grade band, students continue to explore area, perimeter, volume, temperature, and angle measurements.

Additionally, students in grades 1-3 learn to read a clock (analog and digital); in first grade, they tell time to the nearest hour and half hour, in second grade, to the nearest 5 minutes, and in third grade, to the nearest minute. In third grade, students solve problems involving elapsed time. In second grade, students learn about dollars and cents and know all coins and bills by their value.

Reading a Clock

Time can be a difficult subject to teach because it can not be seen or manipulated. Additionally, time is difficult to comprehend for most students because the duration of time depends on what the student is waiting for. Teachers should give students opportunities to time events in their everyday lives, such as brushing their teeth, eating lunch, riding the bus to school, etc.

Benchmarks are critical for learning to tell time. Students need to know that there are 60 minutes in one hour, and that 57 minutes is close to 60. In first grade, limit times to the hour and half-hour.

The Kansas First Grade Flipbook (2017) gives these ideas to support telling time:

- Within a day, the hour hand goes around a clock twice (the hand only moves in one direction)
- When the hour hand points exactly to a number, the time is exactly on the hour
- Time on the hour, i.e., 9:00, is written in the same manner as it appears on a digital clock

- The hour hand moves as time passes, so when it is half way between two numbers it is at the half hour
- There are 60 minutes in one hour; so halfway between an hour, 30 minutes has passed
- Half hour is written with "30" after the colon.

Source: Kansas First Grade Flipbook, 2017

Students need to explore the analog clock. The short hand indicates the approximate time to the nearest hour. The long hand shows minutes before and after the hour. Also notice the clock below; the time is 10:42. It is ten o'clock, even though the short hand is close to eleven. This is confusing for children; one way to talk about this is that the short hand is in the one space. It is not quite eleven o'clock.

Students can connect the idea of a number line to the clock. The number line has been formed into a circle. Another idea is that of the one-handed clock (Van de Walle, Karp, & Bay-Williams, 2019). A one-handed clock helps students focus on the hour hand since the hour hand gives the most information about time. You can purchase an inexpensive clock, set it to the correct time, and remove the minute hand, or you can print a blank clock face and draw in the hour hand. Either way, make sure that the hour hand is in the correct place for a quarter past the hour, a quarter until the hour, half past the hour, and some times that are close to but not on the hour.

Click here for an interactive online analog and digital clock.

Second grade students learn to tell time to the nearest five minutes. There is a good connection here to skip counting by fives. And in third grade, students learn to tell time to the nearest minute, using a.m. and p.m.

Elapsed Time

Solving problems with elapsed time is expected in third grade. Students should use the counting on approach in order to provide a solid foundation. For example, "How much time has elapsed between 8:15 a.m. and 11:45 a.m.?" Students could count on from 8:15 to 11:45 (3 hours) and then 15 minutes to 45 minutes is 30 minutes. So the elapsed time is 3 hours 30 minutes. Or students could draw an open number line.

Money

Students will begin working with money in second grade and solve word problems involving dollars or cents. Second graders have not been introduced to decimals; therefore, ask students to solve problems involving dollars or cents, but not a combination of the two. They will learn about dollars that include $1, $5, $10, $20, $100 bills, and coins that include quarters, dimes, nickels, and pennies. In addition, they should learn to use the $ and ¢ symbols.

Learning the value of each coin can be confusing for young students because the size of the coin doesn't represent the value. For example, think of a dime and a nickel. The dime's value is 10¢ whereas the nickel's value is 5¢, but the nickel is a bigger coin. Therefore, students need to learn the value of each coin by being told, just as they learn the names of other physical objects.

Counting money connects to skip counting by 5s, 10s, and 25s. In addition, introduce "start and stop" counting. This is where students may begin counting by 10s. After several students have counted, tell them to stop and then count by 5s. For example, ask students to count up to 50 by tens (10, 20, 30, 40, 50), then stop and count by fives (55, 60, 65, 70).

Give students multiple opportunities to identify, count, recognize, and use dollar bills and coins. They should also be given the experiences of making equivalent amounts, such as "What are all the possible combinations of coins that equal 47¢?" or "What are all the possible combinations of dollar bills that are equal to 16 dollars?"

Check out this website for some activities involving money.

Length

There is a recommended sequence for teaching measurement. This begins in kindergarten by asking students to make comparisons of measurable attributes, such as longer, shorter, taller, heavier, lighter, etc. This is a critical step in the development of measurement. Give students many opportunities to compare directly so the attribute becomes the focus. Additionally, ask students to discuss and justify their answers to questions such as, “Which box will hold the most?” “Which box will hold the least?” “Will they hold the same amount?” “How could you find out?” Provide students with many items to choose from in order to support their thinking, such as dried beans.

Students then continue by using models of measuring units that produce a number called a **measure**. In kindergarten, start with nonstandard units. For example, ask students, “How many snap cubes tall is the can?” or “How many footprints is the length of this room?” A part of the developmental process in the understanding of measurement is the opportunity to measure.

Students should be given opportunities to make or use their own measuring tools, such as paperclips, or a handprint. After students become proficient in making comparisons and measuring with nonstandard units, you can introduce common measuring tools, such as a ruler.

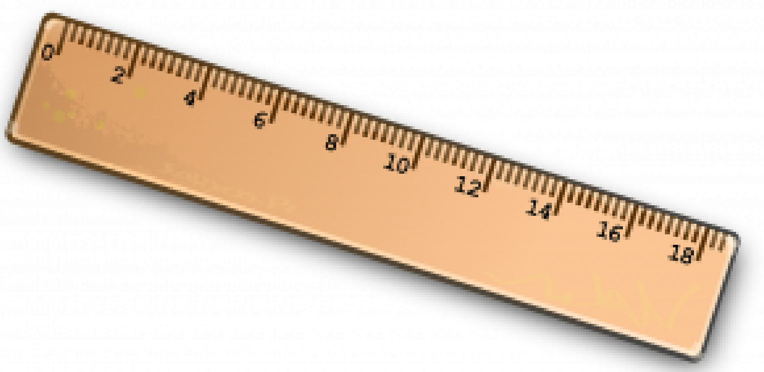
When students make direct comparisons for length, they must notice the starting point of each object and be aware that the objects must be matched up at the end of the object. A developmental milestone for kindergarten students is **conservation of length** which refers to the recognition that moving an object does not change the length.

In first grade, students indirectly measure the length of two objects by using a third object, such as a measuring tool. This is **transitivity** and is connected to conservation. Be sure to use the language taller, shorter, longer, and higher. If students use the words bigger and smaller, ask them to explain what they mean.

When students are measuring an object, they are deciding how many units are needed to fill, cover, or match the object being measured. Ask students to first predict the measurement, then find the measurement, and then discuss the estimates. Additionally, ask students to measure objects that are longer than their measuring tool. This will lead to some good discussions and collaborations among students.

When you ask students to use multiple copies of one object to measure a larger object, this is called **iteration**. Through careful questioning from the teacher, students will recognize the importance of no gaps and overlaps to get a correct measurement.

As students transition from using nonstandard units to standard units to measure, they must be taught to use a ruler correctly when measuring the length of an object. It is critical that students locate the starting point on the ruler. Notice on the ruler below that zero is not at the end.



You can ask students questions such as, “Do you start at the end of the ruler, or at zero?” and “Why do we start at zero?” and “Are we looking at the spaces or the tic marks?” Students need to understand that the spaces indicate the length of the object; the tic marks indicate the end of the space.

Students also need multiple experiences with measuring in inches, feet, centimeters, and meters. Students should measure a length twice and compare the two measurements. For example, the length of the desk is 36 inches or 3 feet; or the length of a paperclip is 3 centimeters or 30 millimeters.

Students should also be expected to estimate lengths using whole units before they measure. Additionally, students should find their own personal benchmark measurements, such as the width of one of their fingers is a centimeter, or the length between their elbow and wrist is a foot.

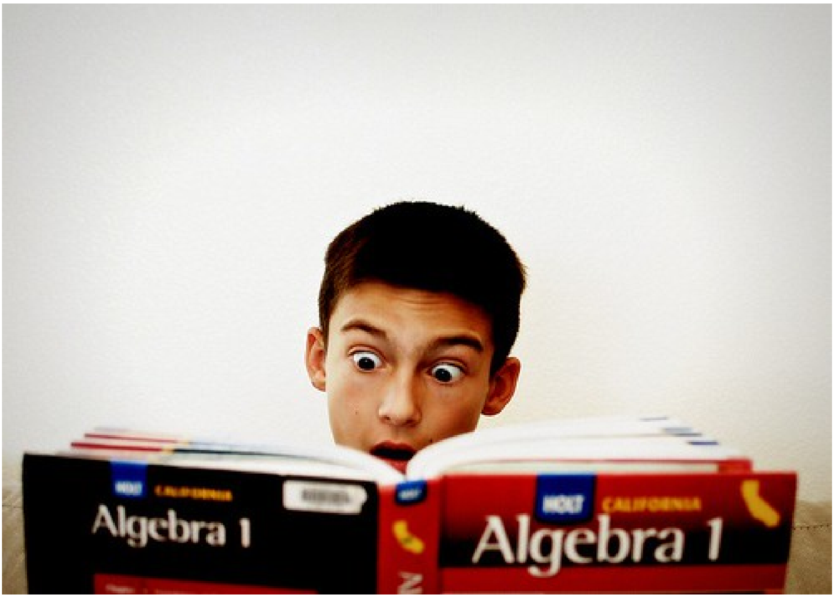
Mass and Volume

Beginning in third grade, students will reason about **mass** and **volume**. Students first measure and estimate the liquid volume and masses of objects using grams, kilograms, and liters. Students need multiple opportunities to weigh objects so they have a basic understanding of the size of weight of a liter, gram, and kilogram. Additionally, students need time to weigh objects and fill containers in order to develop a conceptual understanding of size and weight.

You and your students will find geometry and measurement in almost everything. Studying geometry makes us more aware of the world in which we live and how things are constructed. Geometry should be experienced by children each year from preschool through grade 12. Young children are aware of of spatial relationships and introducing them to geometry helps them realize that mathematics plays an important role in the world in which we live (Egsgard, 1969). Additionally, "geometric reasoning plays a critical role in the development of problem-solving skills, mathematical learning, and reading comprehension” (Schroeter, 2017).

CHAPTER 11

Algebraic Thinking



What is algebraic thinking?

As you read the Kansas Mathematics Standards, you will notice the Domain "Operations and Algebraic Thinking" in all grade levels preK grade 12. Furthermore, algebraic thinking and concepts permeate all areas of mathematics. Algebra is more than manipulating symbols or a set of rules, it is a way of thinking.

According to the K-5 Progression on Counting & Cardinality and Operations & Algebraic Thinking (2011), algebraic thinking begins with early counting and telling how many in a group of objects, and builds to addition, subtraction, multiplication, and division. Operations and Algebraic Thinking is about generalizing arithmetic and representing patterns.

Algebraic thinking includes the ability to recognize patterns, represent relationships, make generalizations, and analyze how things change. In the early grades, students notice, describe, and extend patterns; and they generalize about those patterns. Elementary students use patterns in arrays, and they look at patterns to learn basic facts. According to NCTM Past-President Cathy Seeley, "the development of algebraic thinking is a process, not an event. It is something that can be part of a positive, motivating, enriching school mathematics experience" (Seeley, 2004).

Algebra must be incorporated into the elementary classroom as students see patterns, make generalizations, and move across representations. It is essential that algebra instruction focus on sense making, not symbol manipulation (Battista & Brown, 1998). According to Earnest and Balti, "When elementary teachers are unfamiliar with early algebra, lessons designed and labeled as algebraic may become arithmetic exercises; the algebra then remains hidden from both the teacher and students in the implementation. The result is that the algebra standard is only superficially addressed" (Earnest & Balti, 2008).

Blanton and Kaput (2003) suggest teachers "algebrafy" their current curriculum materials and ask students to discover patterns and make conjectures and generalizations, and justify their answers. They also recommend that teachers use these prompts to extend students' thinking:

- Tell me what you were thinking.
- Did you solve this in a different way?

- How do you know this is true?
- Does this always work?

Therefore, all elementary teachers must support algebraic thinking and create a classroom culture that values "students modeling, exploring, arguing, predicting, conjecturing, and testing their ideas, as well as practicing computational skills" (Blanton & Kaput, 2003).

Key ideas that underscore algebraic thinking are:

- Equality and the concept of equivalence
    - Students often have the misconception that the equal sign means "the answer is" when it really means "the same as." True/false equations are a way to expose students to the meaning of equality, such as 5 + 2 = 3 + 4, or 8 = 2 + 6.
- Inequality
    - Develop in your students a conceptual understanding of greater than and less than as relational symbols, and not rely on memory tricks.
- Positive and negative numbers
    - Students should be exposed to some negative numbers in the early grades. When teachers say, "You can't take 6 from 3," or "You can't subtract a small number minus a big number," teachers are giving students information that just isn't true.
- Problem solving and critical thinking
    - Students who have problem solving and critical thinking skills can solve problems in new contexts and can generalize to new situations.
- Making generalizations

    - It is important that students discover patterns, which includes mathematical rules, in order to make conjectures about the growing pattern.
- Patterns
    - "We need to train children to look for, and to expect to find, patterns in all math work that they do" (Bahr, 2008).
- Variables
    - Variables are unknown and can change and are represented by symbols. Teachers should explicitly explain to students that the one-letter symbol is an abbreviation (Bahr, 2008).
- Relational thinking
    - Relational thinking focuses on the why behind the right answer. For example, 5 x 3 = 15, but why? It is because there are three groups and each group has five.
- Symbolic representation of mathematical ideas
    - Learning that equations communicate the relationship between numbers is crucial for a conceptual understanding behind the symbols.

Source: Algebraic Thinking (de Garcia, 2008)

Connecting Number and Operations and Algebraic Thinking

In kindergarten through grade 3, all of the Operation and Algebraic Thinking standards are related to Number and Operations in Base Ten.

Kindergarten

Students in kindergarten solve addition and subtraction problems in various ways as they make sense of and understand the concepts of addition and subtraction. Kindergarten students should see addition and subtraction equations, and should be encouraged to write equations, such as 4 + 3 = 7 and 7 – 3 = 4, but do not require this of students at this level. It is critical that students see the relationship between numbers, and teachers need to provide students those experiences to manipulate numbers using objects, drawings, mental images, etc. so that students can progress from the concrete, to the pictorial, to the abstract levels.

Kindergarten students work with numbers through 10 as they solve word problems. Be sure to focus on these three problem types: Result Unknown, Change Unknown, and Start Unknown. See Chapter 4 for more information regarding these types of problems as well as Table 1 in the Appendix of the Kansas Mathematics Standards.

Beginning in kindergarten, students should also see the equal sign as a relational symbol, not as an action to find the answer. Help students see the relationship between both sides of an equation as having the same value.

First Grade

All of the first grade standards under the Domain of Operations and Algebraic Thinking were addressed in Chapter 4 as students represent and solve addition and subtraction problems, apply the properties of operations and see the relationship between addition and subtraction, and add and subtract within 20.

Some students misunderstand the meaning of the equal sign, and it is critical to correct this misconception early. The **equal sign** is a relational symbol meaning "the same as." The equal sign is not an operations symbol. Many children believe the equal sign means "the answer is," which is incorrect. Consider the problem 3 + 7 = 9 + 1. There is no answer, although the two sides are the same. Also give students problems where the equal sign is at the beginning of the problem, for example, 8 = 3 + 5.

The Operations and Algebraic Thinking tasks in the Illustrative Mathematics website is a good resource for connecting number sense to algebraic thinking.

Second Grade

All of the second grade standards under the Domain of Operations and Algebraic Thinking were addressed in Chapters 4, 5, and 7 as students represent and solve addition and subtraction problems, use mental strategies to add and subtract within 20, determine even and odd, and use addition to find the total number of objects in a rectangular array.

The common computation situation types teachers should focus on are in Table 1: Common Addition and Subtraction Situations in the Kansas Mathematics Standards. These situation types are result unknown, change unknown, and start unknown. Refer to

"Chapter 9: Whole Number Computation" for more information on these situation types.

As students solve one- and two-step problems, expect them to use manipulatives such as base-ten blocks, a number line, hundreds charts, etc. In second grade, do not focus on traditional algorithms or rules, but instead on the meaning of the operations.

The Operations and Algebraic Thinking tasks in the Illustrative Mathematics website is a good resource for connecting number sense to algebraic thinking. Illustrative Mathematics Grade 2 Operations & Algebraic Thinking tasks

Third Grade

All of the third grade standards under the Domain of Operations and Algebraic Thinking were addressed in Chapters 5 and 7 as students represent and solve multiplication and division problems and understand the relationship between multiplication and division and understand the properties of multiplication. Furthermore, third grade students solve problems involving the four operations and identify and explain the **arithmetic patterns**.

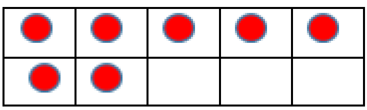
The Operations and Algebraic Thinking tasks in the Illustrative Mathematics website is a good resource for connecting number sense to algebraic thinking. Illustrative Mathematics Grade 3 Operations & Algebraic Thinking tasks

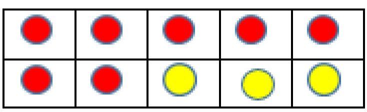
Algebraic Thinking and Algebraic Concepts across the Elementary Grades

Kindergarten

In kindergarten, students identify, duplicate, and extend simple number patterns in preparation for creating rules that describe relationships (NCTM, 2006). Students in kindergarten see addition and subtraction equations as they decompose a set of objects into two sets. Students decompose numbers 1-10 at this level by using objects or drawings. They should record each decomposition by drawing a picture. If students write an equation, they must also share a pictorial representation or show using manipulatives. Additionally, students use the symbols +, -, and = in their decompositions.

Kindergarten students should have multiple opportunities to use Ten-Frames as they "make ten." For example, consider the following three ways that students may solve this problem: *A package has 10 pencils in it. There are 7 pencils in the package. How many pencils are missing?*





Kindergarten students solve problems fluently, which means efficiently, accurately, and flexibly. Accuracy means getting a correct answer, efficiently means solving a problem in a reasonable amount of steps, and flexibly means using strategies such as make ten, counting on, using doubles, using the commutative property, using fact families, etc. Fluency does not mean knowing the answer instantly. Read the paper, "Fluency is More than Speed" by clicking here.

First Grade

First grade students identify, describe, and apply number patterns and properties in order to develop strategies for basic facts (NCTM, 2006). Students in first grade solve addition and subtraction word problems with 20. At this level, do not use letters for the unknown symbols; instead use a box, picture, or a question mark.

$$7 + \square = 9$$

Example: $$\square + 3 = 6$$

In first grade, students are learning to **mathematize** as they model addition and subtraction with objects, fingers, and drawings. This is foundational to algebraic thinking and problem solving. It is critical that students understand the problem situation and represent the problem.

Check out Greg Tang's Word Problem Generator here. You can select the operation, the problem type, the unknown variable, and how many problems to generate.

As students solve word problems, it is critical that they not rely on keywords. When students use keywords when solving a problem, they will strip the numbers from the problem and do not consider the context of the problem. In addition, many "keywords" have multiple meanings, such as altogether and left. The use of a keyword strategy does not develop sense-making and does not build structures for more advanced learning. Instead, discuss what the problem is asking and move the question to the beginning of the problem. Act out the problem, and write the corresponding equation.

Second Grade

In second grade, students use number patterns to extend their knowledge of properties (NCTM, 2006). They represent and solve addition and subtraction problems within 100. Continue to expect students to use base-ten blocks, hundreds charts, number lines, drawings, or equations to support their understanding.

Students in second grade are becoming fluent (efficient, accurate, and flexible) as they mentally add and subtract within

20. Give students multiple opportunities to write equations with two equal addends, such as 2 + 2 = 4 or 3 + 3 = 6. This lays the foundation for multiplication.

Third Grade

Third grade students understand the properties of multiplication and see the relationship between multiplication and division (NCTM, 2006). This is a fundamental step in developing algebraic readiness. Students focus on two models of division: **partition** and **measurement**.

In second grade, students used rectangular arrays to find the total number of objects, and wrote equations to represent the sum, such as a 5 x 5 array could be written as 5 + 5 + 5 + 5 + 5 = 25. See the modules at EngageNY for resources teaching algebraic thinking for third grade students. Students use the properties of multiplication (commutative, associate, and distributive properties) as they multiply and divide, and understand part/part/whole relationships as they make the connections between multiplication and division. To develop algebraic thinking and reasoning, students explain an arithmetic pattern using the properties of operations.

Algebraic thinking is a Domain throughout the mathematics standards. Beginning in kindergarten, students solve addition and subtraction problems by representing them in various ways.

Additionally, they learn about basic operations and quantitative relationships as they model problems and look at mathematical properties and relationships (Stramel, 2021). Read more about Operations and Algebraic Thinking in the K-5 Learning Progressions document.

CHAPTER 12

Data and Data Analysis

Data analysis is all around us – in the news or printed in a newspaper or magazine. We all tend to look at the picture/graph, but not think about the analysis of that data. Therefore, our students have experience with the images related to data, but not the analysis. **Data analysis** is the processing of data in order to find useful information that will help make decisions.

"Data analysis is a body of methods that help to describe facts, detect patterns, develop explanations, and test hypotheses. It is used in all of the sciences. It is used in business, in administration, and in policy" (Levine, 1997). Data analysis provides numerical results; it finds the number that describes a typical value and finds the differences among numbers; it finds averages. But data analysis is not about the numbers – it uses the numbers to answer questions such as "How does it work?"

There are three "rules" of data analysis. First, we look at the data, think about the data, think about the problem, and ask what it is we want to know. Second, we estimate the **measures of central tendency** of the data, and third, we look at exceptions to the central tendency.

Data analysis includes sorting and classifying data, collecting data, and organizing and presenting data. Some examples of data analysis include a tally table, line plot graphs, bar graphs, pictographs, histograms, pie charts, and coordinate grids.

There are two main types of data, **categorical data** and **numerical data**. Categorical data is the type of data that can be put into groups or categories with labels, and describes categories or groups. This is often done according to the characteristics and similarities of the data, such as favorite foods, types of food, or favorite sports. Categorical data can only be placed in one category. Numerical or measurement data is quantitative data that is measurable; such as time, height, weight, amount, etc., or it is a count, such as the number of books a person has.

In kindergarten through grade 5, students mainly focus on categorical data. Specifically, students in kindergarten classify objects into categories; in first grade, students organize and represent categorical data; and in second grade, students draw picture graphs and bar graphs to represent data. Third grade is the most important in the development of categorical data in that students draw picture graphs with each picture representing more than one object, and bar graphs in which the height of each bar is multiplied by a scale factor in order to determine the number of objects in each category.

In grades K-1, students identify and describe measurable attributes. Beginning in second grade, students work with measurement data as they measure lengths. One way to display this data is with a line plot. By the end of fifth grade, students are comfortable with drawing a line plot and analyzing the data so that in sixth grade, students can reason statistically.

Teaching Data and Data Analysis in the Elementary Grades

Students in kindergarten identify and describe the measurable attributes of objects. Students should be given multiple opportunities to describe the attributes. In addition, students should directly compare objects, such as two pencils next to each other in which a student would say, "This pencil is longer." Be sure when students are making direct comparisons that they pay attention to the starting point and see that the two objects must be lined up at the end. At this grade level, students are recognizing and describing the similarities and differences of measurable attributes, such as longer than, shorter, than, taller than, the same size.

As kindergarten students begin to identify the similarities and differences between objects, they use those identified attributes to sort the objects and give the category a name.

The website "Which One Doesn't Belong?" is a resource for teachers and students to help students describe attributes of objects.

First grade students can sort a collection of objects, ask questions about the number of objects in each category, and total the number of objects. Then they can compare the number of objects in each category. Students can create graphs and tally charts of real-world situations, such as favorite ice cream, pets, etc.

Teachers must be sure to ask questions about the data in order to provide a foundation for data analysis. For example, students were asked, “What is your favorite pet? Dogs, cats, or hamsters? Students responded and created the chart below.

What is your favorite pet?	
Dogs	12
Cats	8
Hamsters	3

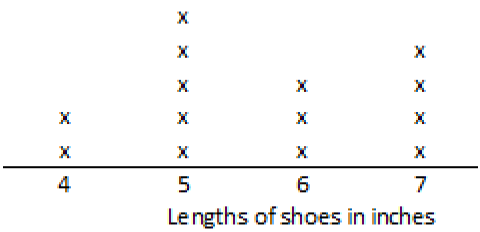
Teachers then need to help students interpret the data. They ask, “What does the data tell us?” and “Does it answer our question?”

- More people like dogs than cats and hamsters.
- Only 3 people liked hamsters as pets.
- 8 people liked cats.
- 4 more people liked dogs than cats.
- The number of people who liked hamsters was 3 less than the number of people who liked cats.
- 23 people answered the question.

In second grade, students represent and interpret data as they measure the lengths of several objects in order to generate measurement data. Students then create a line plot by drawing a number line and then placing an X above the value on the line. Students must be sure to be consistent and line up the Xs so that their line plot resembles a bar graph.

For example, students measure the lengths of their shoes to the nearest whole inch. The class results are on the line plot below.

Number of Shoes Measured



Students also begin to work with categorical data as they organize, represent, and interpret the data. They should draw both pictographs and bar graphs. At this level, pictograph symbols should represent one unit. All pictographs should include a title, categories, category label, key, and data. Bar graphs should be both horizontal and vertical, and be sure to emphasize that all bar graphs require a title, scale, scale label, and category label and data.

In third grade, students read and solve problems using scaled graphs before they draw one.

Representing and interpreting data is a common cluster throughout grades 1-5. In **kindergarten**, students identify and describe measurable attributes of objects, directly compare two objects, and identify the similarities and differences between objects. In **first grade**, students create graphs and tally charts using data that is relevant in their lives, and **second grade** students make line plots, draw picture graphs, and bar graphs to represent a data set. **Third grade** students build on the work they did in second grade and continue to draw picture graphs and bar graphs.

CHAPTER 13

Glossary

Acute triangle – a triangle with all angles measuring less than 90 degrees

Addition symbol – an operation that combines two or more numbers or groups of objects (component parts: addend + addend = sum)

Arithmetic patterns – a pattern that changes by the same rate, such as adding or subtracting the same value each time

Assessment – Conceptual understanding is knowing more than isolated facts and methods; it is understanding mathematical ideas, and having the ability to transfer knowledge into new situations and apply it to new contexts.

Associative Property of Multiplication – when three or more numbers are multiplied, the product is the same regardless of the grouping of the factors

Base ten number system – Our everyday number system is a Base-10 system and has 10 digits to show all numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Cardinal numbers – say how many of something there are

Cardinality – the last number word said when counting, tells how many

Categorical data – a collection of information that can be divided into specific groups, such as favorite color, types of food, favorite sport, etc.

Clusters – groups of related standards

Communication – Mathematics communication is both a means of transmission and a component of what it means to "do" mathematics.

Commutative Property of Addition – numbers can be added in any order and you will still get the same answer

Commutative Property of Multiplication – when two numbers are multiplied, the product is the same regardless of the order of the factors

Computational fluency – using efficient and accurate methods for computing

Connections – the ability to understand how mathematical ideas interconnect and build on one another

Conservation of length – if an object is moved, its length does not change

Contextualize – taking the abstract mathematical representation and putting into context

Data analysis – processing data to find useful information that will help make decisions

Decontextualize – taking a context and representing it abstractly

Defining – attributes that must always be present in order to create that shape

Denominator – how many equal part in the whole amount

Difference – the result of subtraction; the inverse of addition

Distributive Property of Multiplication over Addition – multiply a sum by multiplying each addend separately and then add the products

Dividend – the amount we want to divide up

Divisor – the number we divide by

Domains – larger groups of related standards. Domains are the big idea.

Emergent mathematics – the earliest phase of development of mathematical and spatial concepts

Equal sign – a relational symbol used to indicate equality

Equilateral triangle – a triangle in which all sides are the same length

Equivalence – equivalent fractions have the same value, even though they may look different. For example 12 and 24 are equivalent, because they are both "half"

Expanded notation – writing a number and showing the place value of each digit

Factor – numbers multiplied together

Flexible – the ability to shift among multiple representations of numbers and problem-solving strategies

Fluently – accurately (correct answer), efficiently (within 4-5 seconds), and flexibly (using strategies, such as "making 10" or "breaking apart numbers") finding solutions

Growth mindset – In a growth mindset, people believe that their most basic abilities can be developed through dedication and hard work—brains and talent are just the starting point

Isosceles triangle – at least two sides of the triangle the same length

Iteration – use multiple copies of one object to measure a larger object

Low Floor/High Ceiling Task – a mathematical activity where everyone in the group can begin and then work on at their own level of engagement

Manipulatives – physical objects that are used as teaching tools to engage students in the hands-on learning of mathematics

Mass – a measure of how much matter is in an object

Mathematize – the process of seeing and focusing on the mathematical aspects and ignoring the non mathematical aspects

Measure – to find a number that shows the size or amount of something

Measures of central tendency – The "central value" of two or more numbers. Mean, median, and modes are measures of central tendency.

Measurement – also called repeated subtraction division, a way of understanding division in which you divide an amount into groups of a given size

Measurement division – also called repeated subtraction division, a way of understanding division in which you divide an amount into groups of a given size

Mindset – a person's usual attitude or mental state

Multiplicative Identity Property – the product of any number and zero is zero

Multiplicative reasoning – a recognition and use of grouping in the underlying pattern and structure of our number system

Non-defining – attributes that do not always have to be present in order to create the shape

Numerator – how many parts you have

Numerical data – data that is expressed in numbers rather than word descriptions

Object permanence – understanding that objects exist and events occur in the world independently of one's own actions

Obtuse triangle – a triangle with one angle that measures more than 90 degrees

One-to-one correspondence – numbers correspond to specific quantities

Open number line – A number line that has no numbers. Students fill in the number line based on the problem they are solving.

Ordinal numbers – tell the position of something in the list, such as first, second, third, fourth, fifth, etc

Parallelogram – opposite sides parallel and opposite sides are the same length

Partition – a problem where you know the total number of groups, and are trying to find the number of items in each group

Partitive division – a problem where you know the total number of groups, and are trying to find the number of items in each group

Place value – the value of a digit in its position; for example, the value of the 3 in 236 is 3 tens or 30.

Principles – statements reflecting basic guidelines that are fundamental to a high-quality mathematics education

Problem – any task or activity for which the students have no prescribed or memorized rules or methods, nor is there a perception by students that there is a specific 'correct' solution method

Problem solving – engaging in a task for which the solution method is not known in advance (Bahr & Garcia, 2010)

Product – the result when two or more numbers are multiplied together

Quotient – the answer to a division problem

Rational counting – the ability to assign a number to the objects counted

Reasoning and proof – developing an idea, exploring a phenomena, justify results, and using mathematical conjectures

Rectangle – a quadrilateral with opposite sides parallel and equal length and all angles right angles

Representation – visible or tangible products – such as pictures, diagrams, number lines, graphs, manipulatives, physical models, mathematical expressions, formulas, and equations that represent mathematical ideas or relationships

Rhombus – a parallelogram with all sides equal

Right triangle – a triangle with one 90 degree angle

Rote counting – the ability to say the numbers in order

Round – making a number simpler but keeping its value close to what it was

Scalene triangle – no sides that are the same length

Seriation – the process of putting objects in a series

Spatial sense – an intuition about shapes and the relationships between them

Square – a quadrilateral with opposite sides parallel and all 4 sides equal length and all angles right angles

Standards – define what students should understand and be able to do

Standard notation – writing a number with one digit in each place value

Subitize – visually recognizing the number of items in a small set without counting

Subtraction – an operation that gives the difference or comparison between two numbers (component parts: minuend – subtrahend = difference)

Sum – the result of addition

Technology – calculators, computers, mobile devices like smartphones and tablets, digital cameras, social media platforms and networks, software applications, the Internet, etc.

Transitivity – the ability to indirectly measure objects by comparing the length of two objects by using a third object

Trapezoid – a quadrilateral with opposite sides parallel. The sides that are parallel are called "bases" and the other sides are "legs." Trapezoids are also called trapeziums.

Unit fraction – when a whole is divided into equal parts, a unit fraction is one of those parts. A unit fraction has a numerator of one.

Unitize – a concept that a group of 10 objects is also one ten

Variable – a letter or symbol that stands for a number

Verbal counting – learning a list of number words

Volume – the amount of 3-dimensional space something takes up, or the capacity

Wait time – 20 seconds to 2 minutes for students to make sense of questions

Worthwhile problems – problems that should be intriguing and contain a level of challenge that invites speculation and hard work, and directs students to investigate important mathematical ideas and ways of thinking toward the learning

Appendix

This is where you can add appendices or other back matter.

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