Erich Christian Wittmann

Connecting Mathematics and Mathematics Education

Collected Papers on Mathematics Education as a Design Science

Mathe 2000

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Foreword

One of us, who had the good fortune of meeting Trevor Fletcher in the symposium on the centennial of *L'Enseignement Mathématique* held in Genève in October of 2000, received a piece of excellent advice from this notable mathematics educator, namely, that a fundamental task of the teacher of mathematics is to let students experience the intellectual life that the teacher really lives. Fletcher placed this quote (from E.E. Moise) at the forefront of his paper of 1975 that bears the title "Is the teacher of mathematics a mathematician or not?" [*Schriftenreihe des Instituts für Didaktik der Mathematik Bielefeld*, 6 (1975), 203–218].

Everybody began with a world "without mathematics", this term to be taken with a grain of salt because mathematics is everywhere in our world and comes up frequently and unavoidably in our daily lives, perhaps even without our noticing it. You can imagine such a world by putting yourself in the shoes of an infant who knows no "formal mathematics". Then we gradually move on to another world of mathematics after knowing some elementary mathematics in forming mathematical ideas of objects, notions, theories and techniques out of our experience under some guidance. Then we continue to move on to a world after knowing more "formal mathematics" in refining those mathematical objects, notions, theories and techniques under further guidance. In learning and doing mathematics (which should go hand in hand) the learner proceeds through these worlds and will experience what Fletcher proposed (see Chap. 1 for further elaboration). This is akin to what the esteemed mathematics educator Hans Freudenthal termed as the process of "mathematising".

Freudenthal valued the specificity of subject specific didactics. He believed that the teaching of mathematics could and should only be studied in the perspective of mathematics instead of under any kind of theory of general didactics. In his book *Weeding and Sowing: A Preface to a Science of Mathematics Education* (1978) Freudenthal says, "I see more promise in approaching general didactic problems via the didactics of special teaching areas than in pressing special didactics into the straitjacket of general didactics. It is a priori improbable that a common pattern exists for such different instruction activities as arithmetic and gymnastics." His words vividly describe the shackles that prevent the teaching of mathematics to students with intellectual disability in Hong Kong from making substantive improvement. For decades, institutionalized professional development programmes for teachers teaching these students focus on psychology and general didactics, with subject-specific didactics virtually missing, leaving teachers with the formidable tasks to design effective learning processes that could lead students through small steps directing towards specific achievement. One of the many shocking phenomena is that many students with moderate intellectual disability do not know how to count from one to ten, even at the age of sixteen!

One of us started to be involved in providing mathematics-specific professional development programmes for these teachers six years ago. Core activities include epistemological analysis of content structure and detailed engineering of instructional designs. The former rests solely on the nature of mathematics as an academic discipline, while the latter deals with the design and implementation of learning environments that tailor for specific student groups. For instance, teachers would first examine the structure of counting: (1) remembering the sequence of symbols, (2) remembering the sequence of sounds (pronunciation) for the symbols, (3) matching the quantities with the symbols, and (4) matching the quantities with the sounds. Secondly, teachers are introduced to various tactics to allow students (whose memories are very weak in general) to progressively familiarize and memorize the symbols one by one through carefully designed counting books or counting cards that link up, and provide hints to, the triad of quantity, symbol, and sound. After five years of clinical application, it is now generally observed that students with moderate intellectual disability could progressively learn how to count from one to ten.

Regarding teacher education Freudenthal adopted an integrated approach under which mathematics and its didactics should be handled in an intertwined way. In the same book *Weeding and Sowing: A Preface to a Science of Mathematics Education* he says, "My goal is integrated teacher training, where in particular the subject matter and the didactical component should penetrate each other [...]" What Erich Wittmann has done is to make explicit how this is achieved through (i) the design and execution of substantial learning environments, and (ii) the study of the impacts they bring about. As said in his paper of 2001 [Developing mathematics education in a systemic process, Educational Studies in Mathematics, 48(1), 1–20], "[the] design of substantial learning environments around long-term curricular strands should be placed at the very centre of mathematics education. Research, development and teacher education should be consciously related to them in a systematic way."

The 1960 Nobel Laureate in Physiology or Medicine Peter Medawar says in his book *The Hope of Progress* (1972), "A scientist's present thoughts and actions are of necessity shaped by what others have done and thought before him; they are the wave front of a continuous secular process in which The Past does not have a dignified independent existence of its own. Scientific understanding is the integral of a curve of learning; science therefore in some sense comprehends its history within itself." It reminds us of an intriguing remark, again from Freudenthal in his book *Revisiting Mathematics Education: China Lectures* of 1991, that says, "Children should repeat the learning process of mankind, not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now."

Ever since the early 1980s one of us has been deeply interested in engaging with the integration of history of mathematical developments in the teaching and learning of mathematics. Through such activities one will become aware of the need in examining a topic from three perspectives: a historical perspective, a mathematical perspective, and a didactical perspective. "Although the three are related, they are not the same; what happened in history may not be the most suitable way to go about teaching it, and what is best from a mathematical standpoint may not be so in the classroom and is almost always not the same as what happened in history. However, the three perspectives complement and supplement each other." [M. K. Siu, Study group in history of mathematics—Some HPM activities in Hong Kong, *Education Sciences*, Special Issue, 2014, 56–68.] A teacher of mathematics would do well to know something about the historical perspective, to have a solid idea of the mathematical perspective, and to focus on the didactical perspective. Viewed in this light we note that the history of mathematics means far more than merely anecdotal embellishment in the design of substantial learning environments.

To be a good teacher what matters most is not just how much more the teacher has learnt and knows, nor even how much deeper, but of how differently from various perspectives. In addition, a good teacher should try to carry out what George Pólya maintains that "first and foremost, it should teach those young people to THINK" [On learning, teaching, and learning teaching, *American Mathematical Monthly* 70 (1963), 605–619], and through exploration and thinking to enable students become aware that mathematics makes sense and is thus comprehensible. In this respect the idea of an "elementary mathematics research program of mathematics education" proposed by Erich Wittmann is, in his own words, a "truly interdisciplinary task for which elements of mathematics, its history, its applications, aspects of epistemology, psychology, pedagogy and the mathematics curriculum have to be merged together" [The mathematical training of teachers from the point of view of education. *Journal für Mathematik-Didaktik*, 10, 291–308].

The papers of Wittmann and his colleagues collected in this volume would, on one hand, provide a rich and resourceful collection for enriching teachers of mathematics in this endeavor, and on the other hand, inspire mathematics education researchers who are working towards creating good and great mathematics lessons.

May 2020

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Preface

> *Every sentence, that I write, means always already the whole, therefore always the same, perspectives of the whole, as it were, considered from different angles.*
>
> *Ludwig Wittgenstein*

The invitation by the Japanese Academic Society of Mathematics Education to present my experiences with the design science approach at the annual meeting of this society in 2017 has inspired me to systematically re-think the gradual development of this approach since its inception in the early 1970s.

While working on the written version of this plenary lecture (the last paper of this volume) it occurred to me that it could make sense to combine my major papers on this topic in a book. When I mentioned this idea to some colleagues, they unanimously encouraged me to pursue this and to also include applications of this approach to the developmental research in the project Mathe 2000.

The papers collected in this volume can be classified into four categories:

1. The papers "Teaching Units as the Integrating Core of Mathematics Education" (p. 25), "Mathematics Education as a Design Science" (p. 77), and "Understanding and Organizing Mathematics Education as a Design Science" (p. 265) deal with the *methodological framework* of the approach, and the papers "Developing Mathematics Education in Systemic Process" (p. 191) and "Collective Teaching Experiments: Organizing a Systemic Cooperation Between Reflective Researchers and Reflective Teachers in Mathematics Education" (p. 239) elaborate on the specific status of mathematics education as a *systemic-evolutionary* design science.
2. The *main method for designing learning environments* is described in some generality in the paper "Structure-genetic didactical analyses—empirical research 'of the first kind'" (p. 249). The paper "Designing Teaching: The Pythagorean Theorem" (p. 95), by far the longest of this volume, demonstrates this method with one of the central topics of the curriculum. Structure-genetic didactical analyses are implicit in the many substantial learning environments that are contained in almost all papers, according to the wisdom expressed in the Latin phrase *verba movent, exempla trahunt*.

3. The papers “Clinical Interviews ‘Embedded in the Philosophy of Teaching Units’” (p. 37), “The Mathematical Training of Teachers from the Point of View of Education” (p. 49), and “The Alpha and Omega of Teacher Education: Stimulating Mathematical Activities” (p. 209) deal with *teacher education*. It is worth pointing out that the challenges in teacher education at our university have been the major motivation for conceiving mathematics education as a design science. Over decades the mathematics educators at this university have seen themselves confronted with large courses of up to 700 student teachers *every term*. This has been a real challenge. “Learning environments” have turned out as a very effective tool for introducing student teachers into elementary mathematics, into principles of mathematics teaching, and into the curriculum. On p. 221 an empirical study is mentioned that proves the high acceptance of this approach by student teachers.
4. Four papers exhibit the use of *non-symbolic means of representation* as one crucial element for designing learning environments that are adapted to students’ prior knowledge *and* at the same time are mathematically sound. The paper “Standard Number Representations in Teaching Arithmetic” (p. 161) contains a comprehensive toolkit of non-symbolic representations for a central topic of the curriculum. These representations are so powerful that they are able to carry operative proofs, as shown in the paper “Operative Proofs in School Mathematics and Elementary Mathematics” (p. 223). The difficulties student teachers usually encounter in accepting operative proofs as sound proofs are discussed in the paper “When is a proof a proof?” (p. 61, written in collaboration with G. N. Mueller) that looks at proofs also from a general perspective.

Between the two papers published in 2002 and the paper published in 2014 there is a gap of more than 10 years in which I did not publish any further papers on the design science approach. There is a good reason for this “silence”: during this period my colleague Gerhard Mueller and I concentrated on applying the design science approach to developing the innovative textbook series *Das Zahlenbuch* (K–4). We took this step as we wanted to find out if this approach would work also at the very forefront of the teaching practice. In our textbook work we did not act merely as editors, as is common in the textbook business, but wrote large parts of the student books and the workbooks, all teacher’s manuals, and accompanying materials ourselves, supported by advisory boards of teachers in different parts of Germany. Our intention was to elaborate our conception down to the most specific details. Connected to this work was in-service teacher education on a large scale. One of the mottos of Mathe 2000 is “He who receives criticism should be happy”. It is attributed to the ancient Chinese philosopher Mong Tze. As we communicated this motto to teachers they felt free to speak openly and to provide us with ample and precise feedback from their teaching practice.

As *Das Zahlenbuch* was adapted to other European countries, our close contacts with teachers reached beyond the German borders. A translation into English, *The Book of Numbers*, is used in the Swiss International Schools. The Appendix of the present volume contains pages from this textbook that are related to learning environments described in some papers.

Our experiences in textbook development have convinced us that the design science approach is effective also at this level. The feedback we received from teachers was not only a confirmation of the approach as a whole but also deepened, refined, and extended it.

As an adherent of the genetic principle I do believe that the best way to understand a concept is to see how it originated from a rough idea and how it has been increasingly articulated, expanded, differentiated, and coordinated with other concepts in a continued process. For this reason, I have decided to arrange the papers in the order in which they were written and published. I am convinced that this order will not only facilitate the understanding of the approach, but also stimulate the reader to critically examine this process, to think of variations, extensions, and alternatives. Moreover, the papers in their natural order represent a historical progress of one idea in mathematics education over a quite long period of time. This might be an interesting case study for both experienced mathematics educators and novices. Comments on the papers are given on pp. 20–24.

The objective of mathematics education as a design science is not to design *any* learning environments but rather learning environments that represent mathematical *and* educational *quality* at the same time: "substantial learning environments", as they have been specified. For this purpose, mathematics must be seen not as just a provider of subject matter, but as an *educational task* (Hans Freudenthal). Connecting mathematics and mathematics education requires looking at mathematics from the point of education and also looking at mathematics education with a broad understanding of elementary mathematics. This reciprocal way of thinking, fully addressed in the paper "The Mathematical Training of Teachers from the Point of View of Education" (p. 37), is present in all papers. In the introductory Chap. 1 this important point will be discussed extensively.

I am well aware that in my work I have drawn heavily upon what great minds before have created. For good reasons John Dewey, Johannes Kuehnel, Jean Piaget, Hans Freudenthal, and Heinrich Winter have been chosen as arch fathers of the project Mathe 2000. There is perhaps some merit in systematically applying the design science approach to the developmental research conducted in Mathe 2000 and to teacher education. However, this work has also been greatly influenced by developments in England in the 1960s, the golden age of English mathematics education, I would say, at the Freudenthal Institute Utrecht in the 1970s and 1980s, by Nicolas Rouche's developmental research at the Centre de Recherche sur l'Enseignement des Mathématiques (CREM) in Nivelles/Belgium in the 1980s and 1990s, and by developments in Japan in the same period of time.

The reader will meet some basic quotations, particularly those from John Dewey's works, in several papers. I do not think this as a disadvantage as these quotations deserve to be repeated and as they serve as links between the chapters.

In preparing this volume I have greatly appreciated the cooperation with Natalie Rieborn/Springer Nature, who took care of the editing process, and Barbara Giese/RWTH Aachen who skillfully converted the text into a nice LateX.

I have profited very much from the continuous professional and personal exchange with quite a number of fine colleagues, and I would like to single out five of them. The first is Jerry P. Becker, who I met first at ICME 1 in 1969 and who since then has kept me informed about international developments. As to elementary mathematics and the history of mathematics I owe much to my German colleagues Gerhard N. Müller, co-director of Mathe 2000, Gerd Walther, my first doctoral student and later Professor at the University of Kiel, and my colleagues from Hong Kong, Man Keung Siu and Chun Ip Fung, who by the way are all used to looking beyond their noses.

Last not least, I would like to express my sincere thanks to my Japanese colleagues for the fruitful exchange I have had with them over two decades. I feel solidarity with them in their conscious emphasis on and their commitment (fudoshin) to *teacher education* as I do believe that what ultimately counts in mathematics education is the impact on teachers. The design science approach is subordinated to this end.

I am looking forward to any comments to this volume, and I would be happy to get into contact with mathematics educators and mathematicians who are thinking in similar directions.

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Contents

About the Author

Erich Christian Wittmann studied mathematics and physics at the University of Erlangen from 1959 to 1964 and underwent the practical training for the teaching profession from 1964 to 1966. After finishing his studies with the first and second Bavarian State Examination he joined the Department of Mathematics at the University of Erlangen as a Research Assistant. In 1967 he received the Ph.D. with a dissertation in group theory.

Stimulated and supported by Hans Freudenthal, who he encountered at a conference in 1966, Erich Wittmann's interests turned more and more to mathematics education. After four years in mathematical research he changed to mathematics education, and in 1970 he was appointed full professor of mathematics education at the University of Dortmund, a position he held until his retirement in 2004.

Erich Wittmann has been responsible for teacher education at all levels, including mathematical courses and courses in mathematics education. In his book "Grundfragen des Mathematikunterrichts" [Basic Issues of Mathematics Teaching] published in 1974 he described mathematics education as a design science for the first time. This view has become the foundation of his work.

In 1987 he founded the project Mathe 2000 at the University of Dortmund in cooperation with his colleague Gerhard N. Müller. The main objective of this project has been to capture the beauty and power of genuine mathematics in designing learning environments for all levels, including early mathematics education. Mathe 2000 has influenced the development of mathematics education in Germany and has reached out into the neighbouring countries. In 2010 the early math program developed in the project received the Worlddidac Award.

Erich Ch. Wittmann has also been active in mathematics. For example, he published an elementary solution of Hilbert's third problem, and a paper on "Feynman's lost lecture" about the motion of planets.

Professor Wittmann was a frequent invited speaker at international conferences, including a keynote address at ICME 9 in Japan. In 1998 he was awarded the honorary doctorate by the University of Kiel and in 2013 the Johannes–Kühnel- Prize by the Association for Fostering the Teaching of Mathematics and Science (MNU).

Erich Ch. Wittmann is an avid hiker and likes classical music. He also is an ardent fan of Borussia Dortmund.

Chapter 1
Unfolding the Educational and Practical Resources *Inherent* in Mathematics for Teaching Mathematics

This: Combining thinking and doing
This: Inducing students to combine thinking and doing
is the source point of any productive education.

Friedrich Froebel 1821

The objective of this introductory chapter is to explain the common rationale behind the papers of this volume. The structure is as follows.

The first section shows that learning environments are a natural way to address teachers in their main role, teaching, and that therefore this approach is promising for improving mathematics teaching in an effective way. The section ends with a teaching model based on Guy Brousseau's theory of didactical situations.

The second section illustrates how in a concrete case the general terms in this teaching model can be brought to life by drawing from processes *inherent* in mathematics.

The general principles behind this special case are explained in the third section. It will turn out that they arise from a genetic view of mathematics.

The fourth section deals with the consequences for teacher education that result in demanding special mathematical courses for teachers.

1 From "Instruction and Receptivity" to "Organization and Activity" in Teaching

In 1968 the journal *Educational Studies in Mathematics* (founded by Hans Freudenthal) started with the papers presented at a conference on "How to teach mathematics so as to be useful?" One year later, after two years of teaching at the gymnasium and four years in mathematical research, the present author moved to mathematics education and saw himself confronted with huge challenges in teacher education.

E. C. Wittmann, *Connecting Mathematics and Mathematics Education*,
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This led him to the following question: “How to teach teachers so as to be useful for teaching mathematics so as to be useful?”

The main professional task of a teacher is to prepare, conduct, analyze lessons and mark papers, and the success of teaching crucially depends on getting students actively involved, not by applying extrinsic means of motivation but by applying intrinsic ones. So, it seems a logical decision to develop mathematics education in a way that makes sense at the very front of teaching. This has completely been in line with the position expressed by Richard Elmore (1997):

> “What do I teach on Monday morning?” is the persistent question confronting teachers. Because they are inclined to ask such questions, teachers are often accused by researchers, reformers, and policymakers of being narrow and overly practical in their responses to the big ideas of education reform. Given the state of the current debate on standards-based reform, though, I think the Monday morning question is exactly the right one, and it should be firmly placed in the minds of everyone who purports to engage in that reform.
>
> Consider the following practical issues. Most statements of content and performance standards coming from professionals and policymakers take no account whatsoever of such basic facts as the amount of time teachers and students have in which to cover content. They are merely complex wish lists. In order to be useful in answering the Monday morning question, they have to be drastically pared, simplified, and made operational in the form of lesson plans, materials, and practical ideas about teaching practice.

Mutatis mutandis this statement also refers to present research in mathematics education.

For centuries the professional frame of teachers has been described as the “didactic triad” (Fig. 1) or in a somewhat extended version as the “didactic tetrahedron” (Fig. 2).

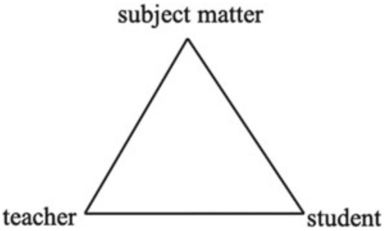

Fig. 1 Didactic triad

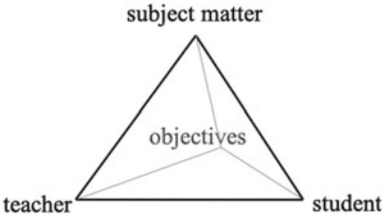

Fig. 2 Didactic tetrahedron

Up to the end of the nineteenth century and into the first decades of the twentieth century the role of the teacher was widely understood as that of an "instructor" or "deliverer of knowledge". His or her task was to separate pieces of subject matter, present it to the students, link it to what they had learned before, embed it into a system and test if the students could reproduce and apply the new knowledge.

The most elaborate form of this view on teaching and learning are the famous "formal stages" by Friedrich Herbart that were elaborated for the teaching practice by his successors. Wilhelm Rein gave them their final form: "Preparation", "Presentation", "Association", "System", and "Application" (for details see de Garmo 2001, Chap. V, 130 ff.).

At the beginning of the twentieth century the "progressive education" movement gave new impetus to voices who had been pleading in favor of a shift for a long time. In 1916 the German mathematics educator Johannes Kuehnel (1865–1928) described the new role of teachers and students according to the new vision in his book "Neubau des Rechenunterrichts" [Re-Installing the Teaching of Arithmetic] (Kuehnel 1954, 69–70, transl. E.Ch.W.) as follows:

> The goal of teaching arithmetic is to provide the students with the foundations for a mathematical penetration of all things and phenomena of nature and human life … When therefore in the enlightened educational view of our time skills appear as certainly indispensable tools and so as an unquestioned objective of teaching, however, not more than tools, it is the task of the future to consciously *replace the mere concentration on skills by true mathematical education*.
>
> The main question that provides the yardstick and the orientation for the whole book can be formulated as follows: *What is the both scientifically and practically founded teaching method by which we can further the development of the student in the desired way?*
>
> This formulation readily reveals the influence of the new orientation. It is not a method by which we want to instruct the student in something in a way as easy, as painless or as pleasurable as possible, be it knowledge or skills. *Instructing, presenting, conveying* are notions of the *art of teaching of the past* and have only little value for the present time; for the educational view of our time is no longer directed to plain subject matter. Of course, the student should acquire knowledge and skills also in future – we even hope more than in the past – however, we do not want to *impose* them on him, but he should *acquire* them himself. In this way also the role of the teacher is changing in every respect. Instead of delivering subject matter he will have to develop the student's abilities. This is something completely different, in particular for teaching arithmetic. For the differently formulated question for the teaching method will deprive the teacher of two instruments that in the past seemed indispensable and as marks of the highest art of teaching: *presenting and forming*. For compensation the teacher gets two other instruments that at first sight seem insignificant, that, however, are much more powerful: *providing opportunities* and *stimulating individual development*.
>
> And the student is no longer tuned to *passively receiving knowledge*, but to *actively acquiring it*. What characterizes the teaching method of the future is not instruction and receptivity, but organization and activity.

In the following decades this view of teaching and learning has gradually spread in many countries and found substantial support from many sides (see, for example, ATM 1967, with a wonderful preface by David Wheeler; Freudenthal 1972[1]; Becker

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[1] Interestingly, Hans Freudenthal had studied Kuehnel's book thoroughly.

and Shimada 1997, translated from the Japanese original published in 1977; Revuz 1980, with a most remarkable title; Winter 1989).

In the early1980s Heinrich Winter, very much influenced by Hans Freudenthal, served as an advisor for a committee engaged with developing a syllabus for the primary school of the state of North Rhine-Westphalia. In the document the role of the teacher is described in Winter's unmistakable style as follows (KM 1985, transl. E.Ch.W.):

> For pursuing the mission and the objectives of mathematics teaching a conception is appropriate in a particular way in which learning mathematics is considered as a constructive, inquiry-based process. This means that students should get as many opportunities as possible for self-reliant learning in all phases of the learning process:
>
> - starting from challenging situations; stimulating the students to observe, to ask questions, to guess
> - displaying a problem or a complex of problems; encouraging students' own ideas and providing support
> - anchoring new knowledge in prior knowledge in manifold ways; summarizing new knowledge as clearly and concisely as possible, in some cases insisting on memorization; encouraging students to practice on their own
> - discussing with students about the nature of the new knowledge and about the processes with which it has been gained (recollection), stimulating students to investigate related problems by themselves.
>
> The role of the teacher consists of finding and offering challenging problems, providing students with conceptually rich teaching aids and productive forms of exercises and above all to establish and maintain a communication that is favorable for the learning processes of all children.

This syllabus also reflected the so-called "general mathematical objectives" that Winter had already formulated ten years previously: Mathematizing, Exploring, Explaining and Communicating (Winter 1975).

Another important innovation brought about by this syllabus is the emphasis on a balanced orientation to both applications *and* structure (applied and pure mathematics) that Winter had postulated in a paper on the role of mathematics for general education, in which he delineated three major objectives of mathematics teaching (Winter 1995):

(1) to perceive and understand phenomena in the world around us that concern us or should concern us, in nature, society and culture, and to do this in a way specific for mathematics,
(2) to get acquainted with mathematical structures, represented in language, symbols, pictures and formulae, and to understand them as mental creations, as a deductively ordered world of its own,
(3) to acquire problem-solving strategies (heuristic strategies) going beyond mathematics by coping with problems.

The design science approach to mathematics education has been born from the intention to assist teachers in these tasks, that is, to provide them with first-hand knowledge for organizing learning processes in the form of elaborated teaching units (later called substantial learning environments). These units should be explicit about how

- to introduce students into mathematical activities by which mathematical knowledge can be acquired,
- to accompany them and to provide support during their activities,
- to assist students in reporting about their observations, in formulating the patterns they have found,
- to assist students in explaining these patterns,
- to fix the knowledge that has been acquired and to summarize it in a pregnant form.

These professional interventions of teachers reflect the natural flow of any goal-directed teaching and learning of mathematics. Guy Brousseau has captured them in five “didactical situations”: instruction, action, formulation, validation, and institutionalization (Brousseau 1997).

Table 1 shows the interplay between the teacher’s interventions and students’ activities whereby italics indicate who is taking the initiative during the situation in question (Wittmann and Müller 2017, 20).

Table 1 Brousseau’s didactical situations

	Instruction	Action	Formulation	Validation	Institutionalization
Teacher	*Explaining the objectives and the problem(s)*, providing students with material	Observing and stimulating students, *if necessary, asking for explanations*	Listening, asking for further explanations	*Stimulating explanations, deepening insights*	*Summarizing the acquired knowledge in a concise form*
Students	Paying attention, listening, asking for further explanations, “joining in”	*Working on the problems, exchanging information with other students*	*Presenting solutions or patterns that have been discovered*	*Explaining solutions and patterns by taking up teacher’s suggestions*	Listening, asking for further explanations

This table is extremely useful for organizing teaching and for analyzing and evaluating lessons along the “Organization and Activity” model of learning and teaching—provided the potential inherent in mathematics is used properly.

The next section illustrates it with an example.

2 The Learning Environment "Calculating with Remainders"

The book *Notes on Primary Mathematics* (ATM 1967) starts with the sketch of a unit ("An Addition Game") that is well suited to show how Table 1 can be brought to mathematical life. In this section this unit will be expanded into a fully-fledged learning environment. In Germany the natural place of this unit in the curriculum is the beginning of grade 5. This grade is traditionally devoted to refreshing knowledge of mental arithmetic, semiformal strategies of calculation, the standard algorithms, and the arithmetical laws from the first four years of education (that in most German states form the primary school).

Objectives: Repetition of arithmetic at the primary level in the context of a mathematical structure that goes beyond the familiar number structure and has applications on the EAN-Number and the ISBN-Number.

Mathematical background: Residue class rings
Teaching materials: Counters, dot arrays, worksheets

1. At the beginning the teacher announces that the following unit is intended to practice arithmetical skills and to explore new mathematical structures that at first sight look a bit strange but give the opportunity for creative work.
2. *Introduction of the tasks*
   First the students are asked to solve some division problems (Fig. 3).
   Based on the results and explanations of the students the teacher emphasizes that any number can be written as a multiple of 10 plus a remainder that is just the Ones digit of the number (Fig. 4)

$$\begin{array}{l} 140 : 10 = \ldots, \\ 143 : 10 = \ldots, \\ 6578 : 10 = \ldots, \\ 7200 : 10 = -\ldots, \\ 19\,561 : 10 = \ldots \end{array}$$

Fig. 3 Divisions by 10 without remainder

$$\begin{array}{l} 140 = 14 \cdot 10 \\ 143 = 14 \cdot 10 + 3 \\ 6578 = 657 \cdot 10 + 8 \\ 7200 = 720 \cdot 10 \\ 19\,561 = 1956 \cdot 10 + 1 \end{array}$$

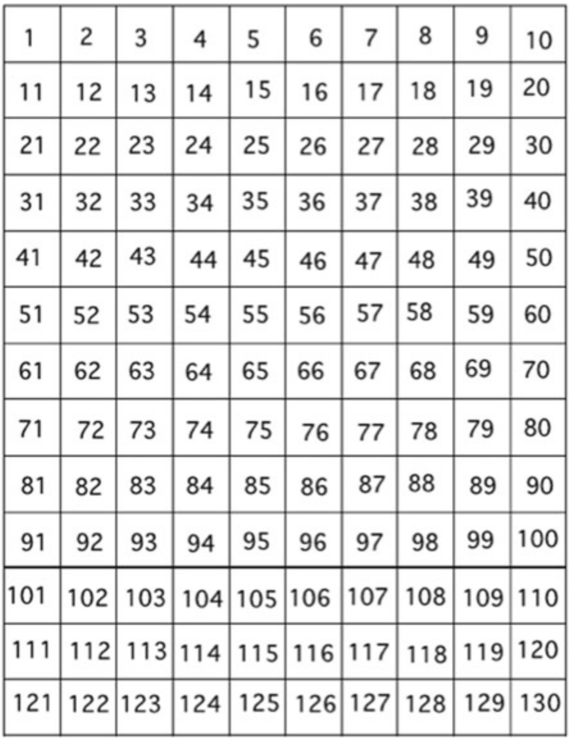
Fig. 4 Divisions by 10 with remainder

The students then receive the extended Hundred chart as a worksheet (Fig. 5) and use it as follows:

Each student chooses two columns of this chart and exerts additions and multiplications only with the numbers of these two selected columns. The teacher explains this rule by means of examples (Fig. 6).

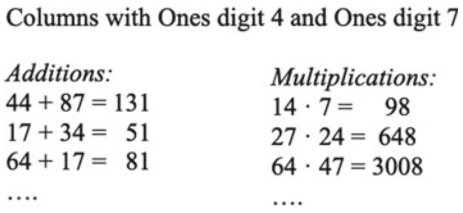
3. *Student work*

   While the students are working the teacher checks if the task has been well understood and provides support where necessary.



1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130

Fig. 5 Extended Hundred chart



Columns with Ones digit 4 and Ones digit 7

Additions:
$44 + 87 = 131$
$17 + 34 = 51$
$64 + 17 = 81$
....

Multiplications:
$14 \cdot 7 = 98$
$27 \cdot 24 = 648$
$64 \cdot 47 = 3008$
....

Fig. 6 Examples of calculations

4. *Report*
   After the students have collected enough data the teacher directs the attention to the Ones digits of the results. The students report on their findings. With the teacher's support they will formulate a pattern: the Ones digits of the results depend only on the Ones digits of the summands resp. the factors.
   For example, the Ones digits 3 and 7 always yield the Ones digit 0 for addition and the Ones digit 1 for multiplication.
5. *Explanation of the pattern*
   The explanation follows immediately from the standard algorithms (Fig. 7):

Fig. 7 Schemes of long addition and long multiplication

   Both the addition and the multiplication of the Ones digits might lead to a carry. This carry and the following calculations do not affect the Ones digits. The size of the numbers does not matter.
6. *Summary*
   The teacher summarizes the findings by telling the students that mathematicians have found it useful to "forget" the tens, hundreds, thousands, etc. and to calculate only with the Ones digits by using slightly different signs:

$$7 \oplus 4 = 1$$

   In words: the Ones digit 7 additively combined with the Ones digit 4 yields the Ones digit 1.
   In short: 7 plus 4 equals 1. In this case, however, "plus" means the new sign $\oplus$.

$$7 \odot 4 = 8$$

   In words: the Ones digit 7 multiplicatively combined with the Ones digit 4 yields the Ones digit 8.
   In short: "7 times 4 equals 8". Again "times" here means the new sign $\odot$.
   The students get a worksheet (Fig. 8) in which some results of the addition and the multiplication table for the Ones have already been entered. The teacher should take time and slowly explain how the tables have to be read and show how sums and products of Ones are entered into the table.
   Only after a thorough clarification should the individual students fill in the missing entries themselves. Of course, students are allowed to cooperate, as always, and to assist each other.

⊕	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0		2			5			8	
1		2		4						0
2	2			5		7	8			1
3			5		7		9	0		
4		5		7						3
5	5		7		9		1		3	
6										
7	7			0				4		6
8			0			3				7
9										

⊙	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0		0							
1		1		3						9
2	0			6		0	2			
3					2		8			7
4				2				8		6
5				5	0			5		
6			2						8	
7		7		1				9		3
8			6			0				2
9				7			4			1

Fig. 8 Addition table and multiplication table modulo 10

7. *Applications*
   Certainly, the students will wonder what purpose these tables are useful for and will be surprised that there is an application in their immediate environment. Both the European Article Numbers (EAN) and the International Standard Book Number (ISBN) consist of 13 digits whereby the last digit is a check digit that is determined in the following way: the first 12 digits are alternately multiplied by 1 and 3 according to the addition and multiplication table for the Ones, and then the sum of the products is determined according to the addition table of the Ones. Finally, the check digit is chosen such that it complements the sum to 0.

Example EAN 978489582586?
First the digits are multiplied alternately with 1 and 3:

$$9 \odot 1 \oplus 7 \odot 3 \oplus 8 \odot 1 \oplus 4 \odot 3 \oplus 8 \odot 1 \oplus 9 \odot 3 \oplus 5 \odot 1 \oplus 8 \odot 3 \oplus 2 \odot 1 \\ \oplus 5 \odot 3 \oplus 8 \odot 1 \oplus 6 \odot 3.$$

From the multiplication table for the Ones we gather the results of the products:

$$9 \oplus 1 \oplus 8 \oplus 2 \oplus 8 \oplus 7 \oplus 5 \oplus 4 \oplus 2 \oplus 5 \oplus 8 \oplus 8.$$

The addition table for the Ones allows us to calculate this sum step by step:

$$9 \oplus 1 = 0, \quad 0 \oplus 8 = 8, \quad 8 \oplus 2 = 0, \ \ldots$$

In shorthand notation:

$9 \oplus 1$	$\oplus 8$	$\oplus 2$	$\oplus 8$	$\oplus 7$	$\oplus 5$	$\oplus 4$	$\oplus 2$	$\oplus 5$	$\oplus 8$	$+ 8.$	
	0	8	0	8	5	0	4	6	1	9	7

The check digit must be 3 as $7 \oplus 3 = 0$.

The teacher explains this procedure by means of examples. Then the students join in to explain the check digits of some other EAN or ISBN Numbers. It is helpful to hand out a sheet with correct addition and multiplication tables for the Ones.

At the end of this unit each student should be able to determine the check digit of the EAN number of an article bought in some shop or check the ISBN number of some of their books.

Possible continuation of this learning environment

The module 10 can be replaced by any module m (see ATM 1967).

It is standard to write the numbers in schemes with m columns. All numbers with the same remainder under the division by m then form a column. Within each column the numbers increase by m. These columns are denoted by the remainders 1, 2, . . . , 0. Figure 9 provides this scheme for the module 5.

First the scheme must be investigated. The students realize that in each column the numbers increase by 5. If a multiple of 5 is added to a number in some column the result lies in the same column. The last column contains the multiples of 5.

If counting along this scheme is accompanied by a growing array of dots the students see that all numbers in a column leave the same remainder when divided by 5 (see the "film" in Fig. 10).

The numbers with the remainder 0 are identified as the multiples of 5 that are known from the multiplication table. In the Mathe 2000 curriculum the multiples of any number are calculated by explicitly using the arithmetical laws. In this case students know already that the sum and the difference of two multiples is again a multiple and that the multiple of the multiple of a number is a multiple of that number, too.

Rem. 1	Rem. 2	Rem. 3	Rem. 4	Rem. 0
1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30
31	32	33	34	35

……………………………………………

Fig. 9 Number table modulo 5

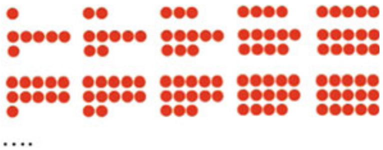


Fig. 10 Iconic representation of the number table modulo 5

In ATM (1967) the scheme in Fig. 8 and similar schemes are investigated. The columns are denoted by letters, A, B, C, D, E etc. As in the Mathe 2000 curriculum "pretty packages" play a prominent role from grade 1, it is natural to tie in with this format. The teacher shows how "pretty packages" can be constructed with the scheme in Fig. 9. Two columns are chosen and then "pretty packages" are constructed by starting with the sum or the product of the two smallest numbers of the columns and increasing one or both numbers gradually. After each calculation the column of the result is noted. Figure 11 shows some examples:

As mentioned previously, the *first objective* of this learning environment is to refresh arithmetical skills. So, it is no disadvantage at all that these calculations take time. The students are free to use mental arithmetic, semiformal strategies, and standard algorithms. They are also free to choose numbers as big or as small as they want.

$3 + 4 = 7$ Rem. 2	$1 + 5 = 6$ Rem. 1	$2 \cdot 4 = 8$ Rem. 3	$1 \cdot 1 = 1$ Rem. 1
$8 + 4 = 12$ Rem. 2	$6 + 5 = 11$ Rem. 1	$7 \cdot 4 = 28$ Rem. 3	$6 \cdot 1 = 6$ Rem. 1
$13 + 4 = 17$ Rem. 2	$11 + 10 = 21$ Rem. 1	$12 \cdot 4 = 48$ Rem. 3	$11 \cdot 6 = 66$ Rem. 1
$13 + 9 = 22$ Rem. 2	$16 + 15 = 31$ Rem. 1	$17 \cdot 9 = 153$ Rem. 3	$11 \cdot 11 = 121$ Rem. 1
$13 + 14 = 27$ Rem. 2	$21 + 15 = 36$ Rem. 1	$17 \cdot 14 = 238$ Rem. 3	$16 \cdot 11 = 176$ Rem. 1
$18 + 14 = 32$ Rem. 2	$26 + 20 = 46$ Rem. 1	$22 \cdot 14 = 308$ Rem. 3	$21 \cdot 16 = 336$ Rem. 1

Fig. 11 Calculations with fixed columns

The most difficult problem is determining the remainder of the results when the numbers get bigger. In the Mathe 2000 curriculum students learn how to decompose a number into multiples of the divisor according to their preferences and they learn the standard algorithm for division, as well as for arbitrary big divisors (see the "smart division" in Wittmann and Müller 2018, 238–240).

Of course, some calculations will be wrong. However, students will quickly guess that all results in a pretty package have the same remainder. This helps them to spot and correct mistakes. After calculating students will have much to report.

Unanimous finding: the remainders of the results do not depend on which numbers of a column have been chosen, but only the columns themselves.

The teacher shows how to fix the findings in tables similar to Fig. 8 (see Fig. 12).

⊕	**0**	**1**	**2**	**3**	**4**
0	0				4
1		2		4	0
2	2				1
3			0		
4			1		3

⊙	**0**	**1**	**2**	**3**	**4**
0	0				
1			2		
2	0			1	3
3			1		2
4			3		1

Fig. 12 Addition table and multiplication table modulo 5

How to explain the pattern? For the module 5 the proof that the columns of the sum and the product of two numbers depend only on the columns of the summands resp. the factors can be given in a way similar to the module 10.

Example The Ones digits 2 and 7 in the column "Remainder 2" and the Ones digits 3 and 8 in the column "Remainder 3" add up to $2 + 3 = 5$, $2 + 8 = 10$, $7 + 3 = 10$, and $7 + 8 = 15$. The results all have the Ones digit 5 or 0. Therefore they belong to the column "Remainder 0".

The Ones digits of products with these Ones digits yield $2 \cdot 3 = 6$, $2 \cdot 8 = 16$, $7 \cdot 3 = 21$, and $7 \cdot 8 = 56$. So, all products have the Ones digits 1 or 6 and belong to the column "Remainder 1". However, it would take time to check all combinations.

The following operative proof uses the arithmetical laws and knowledge from "pretty packages". It has the advantage of covering all modules. For the module 5 the proof runs as follows:

If a summand of a sum or a factor of a product is moved up one step *within a column* it increases by 5. The sum then also increases by 5, which means it stays in the same column. The product increases by a multiple of 5, according to the distributive law. Therefore, the product also does not leave the column.

For the addition these operations can be illustrated by referring directly to counters: if 5 counters are added to a summand then the sum is increased by 5. In the case of multiplication dot arrays render the same service. The increase of a factor of a product by 5 changes the product by a multiple of 5 (see Wittmann and Müller 2017, 71, 202–205).

For grade 5 this proof is quite appropriate, provided that the arithmetical laws have received the attention they deserve from grade 1.

If there is enough time students could investigate other modules (ATM 1967).

The learning environment can be taken up later in the curriculum when the arithmetical laws are available in their formal setting and the operative proof can be re-formulated in the language of algebra. Substantial *mathematics* should be revisited on multiple occasions regardless.

The structure-genetic didactical analysis in this section demonstrates the following points:

1. What has to be taught is determined in broad terms by the syllabus. It is up to the designer to construct learning environments consistent with the syllabus that

   - take up students' prior knowledge
   - present problems that call for students' active participation in investigating these problems
   - provide students with interesting materials for practicing skills and fostering heuristic strategies
   - give students an authentic account of what mathematics is about.
     (see Winter's description of the role of the teacher in Sect. 1)

   It is obvious that *mathematics and its applications are the decisive source* for design.

2. The unit in this section starts with an assignment that leaves the individual student free space for his or her activities. First results are likely to provide feedback for the following calculations. The conjecture (discovery) of patterns leads to the desire for explanations, that is proofs. If earlier in the curriculum the designer has taken measures for introducing tools that are appropriate for formulating a proof then the teacher and the students are well prepared.

   It is obvious that the whole teaching/learning process is essentially determined by the natural flow of a mathematical investigation. Brousseau's didactical situations describe the essential steps in this process. It is *mathematics* that provides the teacher with *first-rate professional knowledge*.
3. The mathematical structure that is carrying the learning environment does not only give stimuli to the teacher, but, as important, also to the students. The more experience students have acquired in past learning, the more they will be able to proceed on their own. Winter's general mathematical objectives "Mathematizing, Exploring, Explaining, Communication" are used all the time. Mathematical activities that give rise to these general mathematical objectives are the best context for learning notations, symbols, termini, expressions, and the informal language that naturally go along with doing mathematics.

   It is obvious that students get *essential stimuli for learning* from *mathematics itself.*

However, the conclusions that are drawn here presuppose a certain view of mathematics. This will be clarified in the following two sections.

3 Mathematics for Specialists and Mathematics for Teachers

From the point of view of the mathematical specialist the structure underlying the learning environment of Sect. 2 is an elementary example of the following general construction:

One starts from the commutative ring $(\mathbb{Z}, +, \cdot)$ of integers in which the following laws hold:

$(\mathbb{Z}, +)$ is a commutative additive group with 0 as the neutral element, the operation $\cdot$ obeys the commutative and associative law, and $+$ and $\cdot$ are connected by the associative law.

The set $m\mathbb{Z}$ of multiples of any number $m > 1$ forms not only a subring of $(\mathbb{Z}, +, \cdot)$, but it also contains all products $s \cdot t$, where $s \in \mathbb{Z}$ and $t \in m\mathbb{Z}$.

Two elements $a, b \in \mathbb{Z}$ are called *equivalent*, denoted by $a \equiv b$, if $a - b \in m\mathbb{Z}$. It is easy to see that this is exactly the case if a and b leave the same remainder when divided by m.

By using the existence of the neutral element 0 for addition, the existence of additive inverse elements, and the associative law for addition, it is proved that the relation $\equiv$ is a reflexive, symmetric, and transitive relation. Therefore, it is an equivalence relation that splits $\mathbb{Z}$ into disjunct equivalence classes. There are m classes, denoted as [0], [1], …, $[m-1]$ according to the possible remainders $0, 1, \dots, m-1$, under the division by m.

The set of these residue classes is called the *residue class ring* $\mathbb{Z}$ modulo m, written as $\mathbb{Z}/m\,\mathbb{Z}$. The elements of a class are called "representatives" of this class. Any of these elements or "representatives" determines the class.

For $\mathbb{Z}/m\mathbb{Z}$ two operations are derived from the operations $+$ and $-$ in $\mathbb{Z}$ as follows: For any classes $[a], [b] \in \mathbb{Z}/m\,\mathbb{Z}$

$$\begin{aligned}&\text{the sum } [a] \oplus [b] \text{ is defined as the class } [a' + b']\\ &\text{the product } [a] \odot [b] \text{ is defined as the class } [a' \cdot b']\end{aligned}$$

where a' is an arbitrary element of $[a]$ and b' is an arbitrary element of $[b]$.

In order to show that these operations are welldefined it has to be proved that the resulting classes are independent of the choice of the representatives.

Proof Assume $a \equiv a'$, that is $a = a' + s \cdot m$, and $b \equiv b'$, that is $b = b' + t \cdot m$.

From the laws holding in $\mathbb{Z}$ we deduce

$$a + b = a' + s \cdot m + b' + t \cdot m = a' + b' + (s + t) \cdot m, \text{ that is}$$

$$(a + b) - (a' + b') = (s + t) \cdot m \in \mathbb{Z}.$$

$$a \cdot b = (a' + s \cdot m) \cdot (b' + t \cdot m) = a' \cdot b' + (a' \cdot t + b' \cdot s + s \cdot t \cdot m) \cdot m, \text{ that is}$$

$$a \cdot b - a' \cdot b' = (a' \cdot t + b' \cdot s + s \cdot t \cdot m) \cdot m \in \mathbb{Z}.$$

By definition we get $a + b \equiv a' + b'$ and $a \cdot b \equiv a' \cdot b'$. The laws holding in $(\mathbb{Z}, +, \cdot)$ are transferred to the structure $(\mathbb{Z}/m\,\mathbb{Z}, \oplus, \odot)$ as follows: [0] is the neutral element of $\oplus$, $[-a]$ the inverse element of $[a]$, the associative law for $\oplus$ follows from the associative law in $(\mathbb{Z}, +)$.

Proof For $a, b, c \in \mathbb{Z}$ we have

$$\begin{aligned}([a] \oplus [b]) \oplus [c] &= ([a + b]) \oplus [c] = ([(a + b) + c] = [a + (b + c)]\\ &= [a] \oplus [b + c] = [a] \oplus ([b] \oplus [c]).\end{aligned}$$

In a similar way the commutative law for $\oplus$ and the commutative and associative law for $\odot$ and the distribute law are derived.

Therefore $(\mathbb{Z}/m\,\mathbb{Z}, +, \cdot)$ is a commutative ring also.

This construction can be generalized to any ring **R** and a subring **I** with the property that for all $s \in \mathbf{R}$ and $t \in \mathbf{I}$ the product $s \cdot t$ belongs to **I**. Such a subring is called an *ideal*. The arising structures are called quotient rings. For the sake of simplicity the signs $+$ and $-$ are also used for $\oplus$ and $\odot$.

In an analogous way quotients of other algebraic structures can be defined.

The construction of quotient structures is a powerful tool of mathematics that marks the turn to modern mathematics at the beginning of the twentieth century. It is far from being easily accessible and needs a long habituation (Gowers et al. 2008, 26):

> Many people find the idea of a quotient somewhat difficult to grasp, but it is of major importance throughout mathematics, which is why it has been discussed in some length here.

In the context of this volume the decisive question is the following: to what extent is it necessary that teachers know the theory of residue class rings *in its mathematical setting*? For mathematical hardliners the answer is clear: teachers need to know it in full it as it is standard in mathematics. For them it is also clear that this knowledge is not only necessary but also sufficient for teaching any decent unit about residue class rings, even if in teaching elementary means have to be employed.

Mathematics educators would hardly agree to the second part of the statement, but many tend to agree to the first part. In Germany the formal theory of residue class rings is a firm part of courses in number theory for student teachers at secondary level and for primary student teachers with mathematics as a major subject. Even books written by mathematics educators follow the formal presentation (see, for example, Padberg 2008).

However, a closer look at this issue leads to a differentiated picture. For teachers who want to teach in the learning environment illustrated in Sect. 3 a knowledge of the theory of residue class rings is certainly necessary, however, not in its formal setting, but in a setting that uses a terminology that is meaningful for communication with students in the early secondary grades. If for a given module the numbers are represented in tables, the equivalence relation is implicitly defined by this scheme, the term "class" is replaced by "numbers in a column", and the remainders are the natural substitutes for the classes. The term "representative of a class" is superfluous. The independence of the operations from the choice of the representatives is secured by an operative proof that uses the laws of arithmetic, however, in a way that students are familiar with (see Sect. 2). The objects of calculations are not classes, but remainders.

This informal treatment of residue classes is mathematically sound and can be expanded into a theory of residue class rings beyond the formal setting (see Sect. 4). For the designer the mathematical theory is nevertheless very important as it displays the logical relationships in a concise way and shows that this structure is no impasse but finds a continuation in many parts of mathematics. It would be stupid to ignore that many learning environments are stimulated by higher mathematics. There is no doubt that the authors of ATM (1967) who designed the first learning environment in their book had full mastery of the formal representation of the theory of residue class rings. So, to know higher mathematics is a significant advantage for the designer. However, it is by far not sufficient for designing learning environments that match students' prior knowledge at various levels.

The tension between informal and formal settings of mathematics visible in this example is a general one. Wolfgang Kroll, one of the most experienced teachers and supervisors in the practical phase of teacher education in Germany, has expressed it as follows (Kroll 1997, transl. E.Ch.W.):

> Mathematics is a network of concepts and theorems that can be knotted in very different ways and captures quite different things: relationships with the real world, views, imaginations, motives, interests, meanings. As a mental activity – so should it be experienced – it also includes the processes that create mathematics. Networks that are built in a linear way according to the scheme "definition, theorem, proof" have a completely different meaning (and function!) in comparison with networks that are knotted according to other needs, with meshes sometimes wider, sometimes closer, sometimes forwards, sometimes backwards, in different colors and for different purposes.
>
> The network knotted at the university is the "scientific system mathematics", the net knotted at school is something different, and could be called "mathematics as symbolic appropriation of reality". Therefore, mathematics at school is neither contained in mathematics at the university nor can it be easily derived from it.

Despite these differences, the network "school mathematics" and the network "university mathematics" must not be seen as contradictory. In several papers (that are referred to in some parts of this volume) John Dewey has dissolved the tension with a *genetic view of mathematics* and by distinguishing between two aspects of mathematics: mathematics as a research field and mathematics as a means for fostering mental growth.

Very illuminating in our context is Dewey's comment on a paper by the geometer G.B. Halsted. This comment appeared in close proximity to Dewey's great paper "The Relation of Theory to Practice in Education". Halsted, the author of a textbook on elementary geometry based on Hilbert's "Grundlagen der Geometrie", had criticized textbooks in geometry for being mathematically incorrect and insisted that textbooks should from the very beginning present "not only the truth, but the whole truth" (Dewey 1903/1977, 218). Dewey fundamentally disagreed and instead suggested to look at learning geometry as a process that advances from more intuitive, context-related, and applied versions of geometry to more rigorous representations (Dewey 1903/1977, 228):

> These two sides, which I venture to term the psychological and the logical, are limits of a continuous movement rather than opposite forces or even independent elements … it is a social wrong under the name of pure science to force [students who are not ready for it] into paths having next to no meaning for them, and which consequently lead next to nowhere.

It is important to understand that both sides are part of *one* mathematics. One side refers to research, the other one to earlier steps in the development of mathematics and to mathematical learning. From the genetic point of view, it is a mistake to consider the mathematics of specialists as the "true" manifestation of this science and to relegate school mathematics to the sidelines as "pre-mathematics". Historically, mathematics itself has developed from elementary theories, and the lower levels have carried the higher ones, not the other way round. In the same way, more elementary levels of learning that are necessarily less advanced carry the higher ones and cannot be skipped. Representations of some mathematics on a higher level cannot be imposed on students who do not have the necessary prior knowledge. It cannot be emphasized enough that mathematics at lower levels is mathematics *in its own right* and that the basic approach to studying mathematics is invariant over the various levels:

> Because mathematics is made by men and exists only in their minds, it must be made or remade in the mind of each person who learns it. In this sense mathematics can only be learnt by being created. We do not believe that a clear distinction can be drawn between the activities of the mathematician inventing new mathematics and the child learning mathematics which is new to him. The child has different resources and different experiences, but both are involved in creative acts (ATM 1967, preface).

Therefore, the best way to organize learning processes is to draw from the potential that is inherent in seeing mathematics as a growing organism, which requires a special approach. At a conference dedicated to clarifying the scientific status of mathematics education (or didactics of mathematics) in 1975 Trevor Fletcher answered the question of whether the teacher of mathematics is a mathematician or not in the following way (Fletcher 1975, 217):

> I have come to the conclusion that the teacher of mathematics certainly needs to be a mathematician, and that he needs to be a special sort of mathematician. He needs the general mathematical background that enables him to talk on equal terms with mathematics graduates, although he does not need some of the more specialized areas of mathematics that form part of most degree courses which are devoted exclusively to the subject. He needs a broad knowledge of applications in the world outside and in other parts of the school curriculum.
>
> In addition, the teacher needs specialist skills of his own, in the translation of mathematics from one form into another, in understanding the pattern of thinking of his pupils at various stages of development, and in understanding the relevance of structural ideas in mathematics to the teaching of it.

Mathematics educators who want to serve teachers at the very front of teaching must be mathematicians of the special sort described by Fletcher.

The conclusion also from this section is that *mathematics* if seen in its development is the *most valuable source* for designing learning environments and curricula and for providing teachers with basic knowledge for preparing and conducting lessons and for analyzing learning processes. In fact, it is the *natural source*, stands for mathematical authenticity, and is the only way to preserve the beauty of mathematics.

4 From “Instruction and Receptivity” to “Organization and Activity” in Teacher Education

The quotation at the beginning of this chapter consists of two parts. Its meaning is nicely supported by Heinrich Froebel’s introduction into his third geometric gift: a cube divided into eight smaller cubes. Froebel suggests to the nursery school teacher to play a double role: first she should take this material into a room where nobody can disturb her and work intensively with this material herself *as a learner* in order to get thoroughly acquainted with it. Only then will she be in a position to slip into her role *as a teacher* and work with children.

In the same sense mathematics teachers should first explore a learning environment *on their own* without thinking about teaching. Only after they have become familiar with it they should think about how to guide students in exploring it. This access would greatly strengthen the familiarity with mental processes as fixed in Winter’s general objectives.

The mathematical education of teachers should support this approach. In his paper on “The Relationship of Theory and Practice in Education” a long chapter is devoted to the training in subject matter, in which Dewey is explicit in this respect (Dewey 1903/1977, 263):

> Now the body of knowledge which constitutes the subject-matter of the student teacher must, by the nature of the case, be organized subject-matter. It is not a miscellaneous heap of separate scraps. (…) There is, therefore, method in subject-matter itself … method of the highest order which the human mind has yet evolved, scientific method (…) Such being the case, there is something wrong in the “academic” side of professional training, if by means of it the student does not constantly get object-lessons of the finest type in the kind of mental activity which characterizes mental growth, and hence the educative process.

It should be obvious that the typical formal treatment of residue class rings that is suitable for specialists does not meet these requirements as it follows the paradigm “Instruction and Receptivity”, excludes mathematical processes, and leaves little room for student teachers’ activities.

What is needed instead is an introduction into this topic that is closer to the learning environment in Sect. 3. In this context the derivation of the laws in $Z/m\,\mathbb{Z}$ from the laws in $\mathbb{Z}$ is quite easy:

The remainder 0 is the neutral element in $Z/m\,\mathbb{Z}$ and its own inverse, for remainders $a \neq 0$ the remainder $m - a$ is the inverse.

Proof of the associative law for $\oplus$ in $Z/m\,\mathbb{Z}$:

Let a, b, c be remainders module m. The result of $(a \oplus b) \oplus c$ lies in the same column as the result of $(a + b) + c$ and the result of $a \oplus (b \oplus c)$ lies in the same column as the result of $a + (b + c)$. We have $(a + b) + c = a + (b + c)$ because of the associative law in $\mathbb{Z}$. Therefore,

$$(a \oplus b) \oplus c = a \oplus (b \oplus c).$$

In the same way the commutative law for $\oplus$, the commutative and associative law for $\odot$, and the distributive law for $\oplus$ and $\odot$ can be derived.

Within this setting student teachers are given free space to determine the addition and multiplication tables for various modules, to compare the multiplication tables, and to discover similarities and differences. They could, for example, discover that for some modules all products of a remainder $\neq 0$ with the other remainders are different, while for other modules there are remainders $\neq 0$ for which the product is 0, for example, in the case of the module 10, where $5 \odot 2 = 0$. In a formal treatment these phenomena would be done away by a dry proof of the theorem that in case of a module m that is a prime number the ring $(\mathbb{Z}/m\,\mathbb{Z}, \oplus, \odot)$ is a field.

In a course for teachers it would make sense to connect this topic to place value systems with bases different from 10. This again would open up room for investigations. For the base 5-system the proof of the dependence of sums or products only from the columns would work exactly as in the base 10-system. And this could be transferred to other place value systems.

For special multiplication tables the students could investigate which remainders are squares of others: this would give the teacher a chance to mention the research by Gauss and others about the so-called reciprocity law.

In a course of this kind it would also make sense to look at examples where knowledge about residue class rings is applied for solving problems. Elementary number theory is full of such examples that are an important enrichment of the theory.

Mathematical courses that are related to teaching are highly accepted by student teachers. They help them to see, appreciate, and use the educational and practical resources inherent in mathematics. Such courses also contribute greatly to developing a positive attitude towards the subject. For this reason, they are the key to real progress in mathematics teaching, and this applies to all levels. The design science approach matters to teacher education also.

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Comments and Some Personal Remarks on the Papers in Chapters 2–14

In addition to the general background described in this chapter it seems appropriate to say a few words about the origin of each paper and the context in which it was written.

Chapter 2: Teaching Units as the Integrating Core of Mathematics Education. *Educational Studies in Mathematics* 15 (1984), 25–36.

This paper is a translation of the German original

> Unterrichtsbeispiele als integrierender Kern der Mathematikdidaktik.
> *Journal für Mathematik-Didaktik* 3 (1982), 1–18.

that was based on a plenary lecture in Darmstadt in 1981 at the annual meeting of mathematics educators from German-speaking countries.

It was not the first paper of mine on the design science approach. In 1974 there was a lively discussion on the scientific status of mathematics education to which I had contributed the article "Didaktik der Mathematik als Ingenieurwissenschaft" ["Didactics of Mathematics as Engineering"], Zentralblatt für Didaktik der Mathematik 74/3, 119–121. This article was just a first attempt in this direction. The later paper of 1984 presented the idea more clearly, even if the terminology was still in a preliminary state, as well as the diagram (Fig. 1).

Chapter 3: Clinical Interviews Embedded in the "Philosophy of Teaching Units"—A Means of Developing Teachers' Attitudes and Skills. In: Christiansen, B. (ed.), *Systematic Cooperation Between Theory and Practice in Mathematics Education*, Mini-Conference at ICME 5, Adelaide 1984. Copenhagen: Royal Danish School of Education, Dept. of Mathematics 1985, pp. 18–31.

This paper was inspired by a sabbatical leave in Switzerland in 1974 where I had a chance to attend Jean Piaget's Monday seminars. Although at that time my main interest was devoted to the formalization of Piaget's concept of grouping, I noticed that the research method used by the members of the Centre d'épistémologie génétique was very well suited for developing the attitudes that characterize a good teacher. The paper gives an account of a course that right after my return from Geneva was firmly integrated into our teacher education program. The course consisted of two parts: in part 1 the basics of Jean Piaget's genetic epistemology were given (see Wittmann, E.Ch., Mathematisches Denken im Vor- und Grundschulalter [The Development of Mathematical Thinking at the Pre-school and primary level]. Wiesbaden. Vieweg 1982); in part 2 the student teachers conducted clinical interviews, documented them, and presented the results in a seminar. In retrospect the participants evaluated this course highest in the whole program. Again, the terminology in this paper is preliminary. What now is called the "design science approach" appears in the paper (and already in the previous paper) as "Philosophy of Teaching Units".

Chapter 4: The Mathematical Training of Teachers from the Point of View of Education. Journal für Mathematik-Didaktik 10 (1989), 291–308.

My first years in teacher education coincided with the heyday of "New Math", a movement that I had rejected from the very beginning. The pressure at that time to be "mathematically correct" was enormous. It was not easy to get rid of the formats of courses that I had been living with for a decade during my study of mathematics and my years in a department of mathematics. I consider my book *Elementargeometrie und Wirklichkeit* [*Elementary Geometry and Reality*] (Wiesbaden: Vieweg 1987), as a personal breakthrough that was reflected in Chap. 4, originally presented as a survey lecture at ICME 6, Budapest 1988. When preparing this paper, it occurred to me that informal representations are crucial for unfolding the resources for teaching that are inherent in mathematics. The geometry book was also an attempt to mix structure and applications in mathematical courses.

Chapter 5: When is a proof a proof? *Bulletin de la Sociétè Mathématique de Belgique*, Série A, Tome XLII (1990), 1542.

The German original

> Wann ist ein Beweis ein Beweis? In: Bender, P. (Hrsg.), *Mathematikdidaktik: Theorie und Praxis*. Festschrift für Heinrich Winter. Berlin 1988, S 237–257

was written in collaboration with Gerhard Mueller and dedicated to Heinrich Winter on the occasion of his sixtieth birthday.

The paper has been another decisive step in getting rid of the rigid bonds of formal mathematics. The documents included in the paper show that it has not been easy for student teachers to accept informal proofs as they had been "drilled" in formal presentations at school. In 1988 I attended a conference at the island of Samos, the birthplace of Pythagoras, and presented this paper. In the very first row, Andrew Gleason, one of the four mathematicians who contributed to solving Hilbert's fifth problem, was carefully listening to my talk. As he didn't say anything in the discussion, I asked him afterwards if he would agree to these proofs, to which he answered: "Of course, why do you ask me?". One year later, when I presented the same paper at a conference of the German Association of Teachers of Mathematics and Science (MNU) in Darmstadt many participants openly disagreed with my message. In the last mail I received from Hans Freudenthal in spring 1990 he complained of the stubbornness of German gymnasium teachers (my former colleagues!).

Chapter 6: Mathematics Education as a "Design Science". *Educational Studies in Mathematics* 29 (1995), 355–374.

In 1987 Gerhard Mueller and I had collected enough experience in our courses in mathematics and mathematics education for student teachers and thought it appropriate to found the project Mathe 2000 and to explicitly base it on the design science approach. After finishing our work on the two volumes of the *Handbuch produktiver Rechenübungen* [Handbook for practicing skills in a productive way], published in 1990 and 1992, it seemed about time to re-assure ourselves about our approach. The result was the paper

Mathematikdidaktik als "design science". *Journal für Mathematik-Didaktik* 13 (1992), 55–70.

When I was invited for a plenary presentation at the ICMI-Study Conference on "Mathematics Education as a Research Domain. A Search for Identity", chaired by Anna Sierpinska and Jeremy Kilpatrick, in Washington, D.C. in 1994, I presented the English translation of this paper that was also published in the ICMI-Study.

In comparison with the paper in Chap. 2, this version of the design science approach is much more differentiated in many respects. Above all, mathematics education is described as a "*systemic-evolutionary*" design science surrounded by a series of related disciplines (see Fig. 1). There is also progress in seemingly tiny things. For example, the original representation of arithmogons (Fig. 2) had been changed (Fig. 2) in order to facilitate the use of counters. However, the term "teaching units" was retained.

From that time on, my courses on mathematics education took on the following format. In the first 45 min of a lecture I presented five teaching units on a topic that incorporated a theoretical principle, and in the second 45 min I explained this principle. In a follow-up seminar (1.5 h) the participants were given worksheets that stimulated activities for deepening their understanding of the topic. The acceptance of these courses by student teachers was very high, and I cannot but recommend the format.

Chapter 7: Designing Teaching: The Pythagorean Theorem. In: Cooney, Th. P. (ed.), *Mathematics, Pedagogy, and Secondary Teacher Education*. Portsmouth, NH: Heineman 1996, 97–165.

In the early 1990s my colleague Georg Schrage and I were invited by Tom Cooney, at that time professor at one of the leading American centers of mathematics education in Athens/Ga., to join an NSF project that was aimed at bridging the gap between mathematics, mathematics education, and the teaching practice. This project gave me a chance to demonstrate the main method of the design science approach, later called structure-genetic didactical analysis, in some detail, including mathematical analyses, clinical interviews, the design of lessons, and theoretical considerations.

The style of the paper differs from that of the other papers due to the objective of the NSF project: to include tasks that can directly be used in teacher education. As teacher education is a focus in this volume, the style has been maintained.

Chapter 8: Standard Number Representations in Teaching Arithmetic. *Journal für Mathematik-Didaktik* 19 (1998) 2/3, 149–178.

While working on the "Handbuch produktiver Rechenübungen", a grammar of non-symbolic representations for arithmetic had been developing almost by itself. The paper in Chap. 8 is the summary of our experiences during this work. In the new edition of the "Handbuch" (2017/2018) the conception is carried a step further and connected with the notion of operative proof in a consistent way. The recent software "Plättchen & Co. digital" [Counters & Co. digitally] contains digital versions of most of the teaching aids listed in this paper.

Chapter 9: Developing mathematics education in a systemic process. *Educational Studies in Mathematics* 48 (2002), 1–20.

In this paper, a plenary lecture at ICME 9, the design science approach was further elaborated with respect to systemic boundary conditions. One section is devoted to teacher education.

Chapter 10: The Alpha and Omega of Teacher Education: Stimulating Mathematical Activities. In: Holton, D., *Teaching and Learning at University Level.* An ICMI Study. Dordrecht: Kluwer Academic Publishers 2002, 539–552.

While in Chap. 4 a new conception of mathematical courses for teachers is sketched in broad lines, the paper in Chap. 10 describes a format for these courses in detail. Similar to my courses on mathematics education, this format also includes two parts: the first organizes student teachers' activities on a series of topics, and the second includes a systematic presentation of these topics.

As mentioned in the paper, a group of mathematics educators has joined together in writing a book, "Arithmetik als Prozess" [Arithmetic as a Process], based on this format. This book was intended as the first volume of a series, "Elementary Mathematics as a Process", that, however, has not been continued to date. I think it would be promising to restart this project at the international level as a joint venture of mathematicians and mathematics educators.

Chapter 11: Operative Proofs in School Mathematics and Elementary Mathematics.

Translated from the German original

> Operative Beweise in der Schul- und Elementarmathematik. *mathematica didactica* 37, H. 2 (2014), 213–232.

Operative proofs that are inspired by the "operative principle" have been cultivated by quite a number of German mathematics educators. This type of proof is already discussed in some of the previous papers, and in some length in Chap. 7. In Chap. 11 the notion of operative proof is elaborated on in some detail and illustrated by typical examples. In the new edition of the "Handbuch" 2017/2018 operative proofs are developed systematically from grade 1, in line with standard representations.

Chapter 12: Collective Teaching Experiments: Organizing a Systemic Cooperation Between Reflective Researchers and Reflective Teachers in Mathematics Education. In: Nührenbörger, M. et al. (2016). Design Science and its Importance in the German Mathematics Educational Discussion. (S. 26–34) Rotterdam: Springer Nature.

This short paper presented at ICME 13 expands on the systemic constraints of teaching by essentially referring to the work of Donald Schön. The paper introduces the idea of "collective teaching experiments" stimulated by the Japanese lesson studies. In the new "Handbuch" 2017/2018 proposals for such experiments are made for teaching arithmetic.

Chapter 13: Structure-genetic Didactical Analyses—Empirical Research “of the First Kind”. In: Błaszczyk, P. & Pieronkiewicz, B. (eds.), Mathematical Transgressions 2018. Kraków: Universitas, 133–150.

This paper, a plenary paper presented at a conference in Cracow in 2015, describes the main research method of the design science approach, the structure-genetic didactical analysis, and shows that this method also helps to disclose empirical evidence about the feasibility of learning environments. The second part of the title is in no sense intended as a provocation. It simply states that mathematics educators have always included experiences from their teaching practice or that of teachers they were in contact with.

Chapter 14: Understanding and Organizing Mathematics Education as a Design Science. Origins and New Developments. *Hiroshima Journal of Mathematics Education* 12 (2019), 1–20.

This paper is based on a plenary lecture at the annual conference of the Japanese Academic Society of Mathematics Education, Hiroshima 2017. All aspects that are discussed in this chapter are included, and in many places references are made to the new “Handbuch” 2017/2018 and to the “Book of Numbers”. In Fig. 1 the related disciplines have been re-ordered. The disciplines on the left side are more closely connected to mathematics education than those on the right side. New in the picture, on the left side, is “semiotics”. This is due to Willi Doerfler’s epochal work in this area. His paper “Wieso kann man mit abstrakten Objekten rechnen? [How is it possible to calculate with abstract objects?]”, quoted in Chap. 8, has been an eye-opener for me.

Erich Ch. Wittmann

Chapter 2
Teaching Units as the Integrating Core of Mathematics Education

Abstract How to integrate mathematics, psychology, pedagogy and practical teaching within the didactics of mathematics in order to get unified specific theories and conceptions of mathematics teaching? This problem—relevant for theoretical and empirical studies in mathematics education as well as for teacher training—is considered in the present paper. The author suggests an approach which is based on teaching units (Unterrichtsbeispiele). Suitable teaching units incorporate mathematical, pedagogical, psychological and practical aspects in a natural way and therefore they are a unique tool for integration. It is the aim of the present paper to describe an approach to bridging the often deplored gap between didactics of mathematics teaching on one hand and teaching practice, mathematics, psychology, and pedagogy on the other hand. In doing so I relate the various aspects of mathematics education to one another. My interest is equally directed to teacher training and to the methodology of research in mathematics education. The structure of the paper is as follows. First I would like to make reference to and characterize an earlier discussion on the status and role of mathematics education; secondly, I will talk about problems of integration which naturally arise when mathematics education is viewed as an interdisciplinary field of study. The fourth and essential section will show how to tackle these problems by means of teaching units. The present approach is based on a certain conception of mathematics teaching which is necessary for appreciating Sect. 4. This conception is therefore explained in Sect. 3.

1 Discussion of the Status and Role of Mathematics Education

In spite of the important progress in international cooperation on mathematics education achieved during the last decade, discussions on the specific quality of mathematics education (or didactics of mathematics) were mainly restricted to the national level. As for the situation in Germany it was only at the 5th Annual Meeting on

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This paper is a modified version of the opening address given by the author at the 14th Annual Meeting of German Mathematics Educators, Darmstadt, March 1981.

E. C. Wittmann, *Connecting Mathematics and Mathematics Education*,
https://doi.org/10.1007/978-3-030-61570-3_2

Mathematics Education (1971) in Bayreuth that H. G. Steiner stimulated a broad discussion on this issue leading to a series of papers on the status and role of mathematics education (Bigalke et al. 1974).

The picture of mathematics education emerging in this way was not at all homogeneous—and it is still not so today. However, restricting attention to the views of those authors who work as professional mathematics educators in research as well as in teacher training reveals a remarkable agreement: mathematics education (didactics of mathematics) is considered as a *discipline of its own* related intrinsically to mathematics, psychology, pedagogy, and some other fields of study, as well as to the practice of mathematics teaching (cf. Fig. 1).

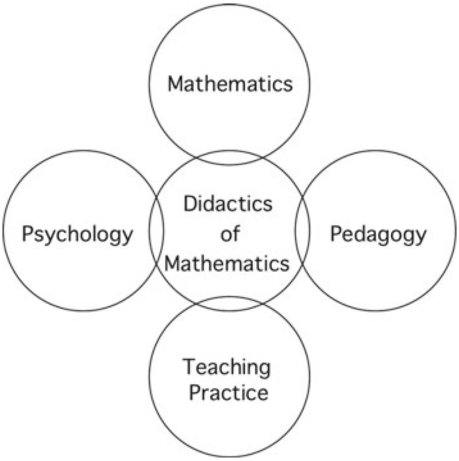

Fig. 1 Didactics of mathematics and the surrounding disciplines

Corresponding to Fig. 1 the memorandum of the "Gesellschaft für Didaktik der Mathematik" on teacher training (March 1981) has described mathematics education as follows:

> It is the task of the teacher to combine mathematical knowledge, psychological experience, and a positive attitude towards young people in order to stimulate and support the learning of students towards educational goals beyond mathematics. The teacher's thinking and doing is characterized by taking into account the interlacing of mathematical, pedagogical, psychological and practical conditions and by making balanced decisions. Accordingly, a reasonable training of mathematics teachers has to include didactical studies in addition to solid mathematical and educational studies.
>
> It is the didactics of mathematics which in an interdisciplinary manner investigates the complexity of mathematical learning and teaching. As *the* professional field of study of mathematics teachers it has to introduce the student teachers to the integrative view and practice which are necessary for their profession and to explain the meaning of their educational work in mathematics. The didactics of mathematics relates mathematical and educational studies to one another, and it provides the necessary bridge to teaching practice.

In short, the feature of mathematics education (didactics of mathematics) expressed in this quotation can be described by the catchwords "interdisciplinary", "integrative" and "applied". Obviously these requirements are the immediate consequences of the complex working conditions of mathematics teachers.

2 Problems of Integration

As we all know it is much easier to postulate interdisciplinarity, integration and applicability than to put them into practice. Apart from inherent issues, we also have to take into account external pressures which push mathematics education in certain directions contrary to interdisciplinarity and integration. Very clear indicators for pressures of this kind are the quite different expectations about mathematics education expressed by specialists of related fields. For example, mathematicians, if they admit the necessity of mathematics education at all, very often consider it to be elementary mathematics. They expect the didactician to be qualified by standards of mathematical research and to keep mathematically alive, at least by working in "a small mathematical garden" (H. Meschkowski). Sometimes practical experience is called for, by which is usually meant naive experience. Possible sympathies of mathematics educators for psychology or pedagogy, however, are criticized or even rejected on the grounds that they lead away from mathematics. On the other side most pedagogues and psychologists regard the didactics of mathematics as part of the disciplines of education. Affinity for mathematics arouses suspicion and quickly qualifies the mathematics educator as a narrow-minded specialist of mathematics. What are the expectations of *practicing* teachers for mathematics educators? In their eyes mathematics educators should have about 10 years teaching practice and should continuously be involved in school life, at best by part-time teaching. The theoretical investigations carried out by mathematics educators are conceived of as unnecessary, if not obnoxious, digressions.

Tensions between didacticians of mathematics and academians in the reference fields arising from these role expectations are sometimes very hard to endure, particularly when the mathematics educators form no department of their own but are integrated into the departments of mathematics or education. Nevertheless, I do not recommend a surrender to the temptation of reducing the tensions by one-sided adaptation. This would inevitably widen the gap to the other fields of reference and would invalidate the tasks of mathematics education. Frankly speaking I think it a sign of weakness. It should be beneath the didactician's dignity to adapt to the environment like a chameleon.

Instead I would like to argue strongly in favour of an independent didactics of mathematics, and I consider the problems of relating the didactics of mathematics to mathematics, to the educational disciplines, and to teaching practice as completely natural problems which should be made conscious, in order to stimulate the mutual crossings of borders. I am deeply convinced that in the long run this will be profitable not only for mathematics education but also for the fields of reference.

Progress in solving the problems of integration mentioned above is of particular importance for teacher training, as most teacher training programmes consist of isolated mathematical, educational, didactical and practical components. Also, research in mathematics education very often lacks the interlocking of different aspects.

The following approach to integrating the various aspects of mathematics education originated in the reform of our teacher training programs at Dortmund University initiated by the 1976 teacher training law of North Rhine-Westphalia. Actually I will restrict myself to teacher training for the primary level, because in this area we have made the greatest progress; however, I am convinced that our approach can easily be transferred to the other levels.

Of course we are not the first ones trying to reform teacher training by integrating different components, neither are we the first ones to use teaching units for this purpose. Therefore I do not believe the approach of this paper to be totally new; however, I think it worthwhile to elaborate the full momentum of the "philosophy of teaching units" in a systematic and comprehensive way. The main part of this paper will be devoted to that task.

3 Some Views on Mathematics Teaching

A more detailed analysis of the problems of integration in Sect. 2 leads very quickly to uncovering an inconsiderate use of language in discussions on teacher education which is responsible for obscurities and misunderstandings, as well as for seeming consensus: the terms "mathematics", "psychology", "pedagogy", "mathematics education", "teaching practice", etc., are tacitly assumed to have a definite meaning, although quite to the contrary, all these fields more or less obviously admit of varieties of very differently marked points of view.

With respect to this plurality of views it is hopeless from the very beginning to establish relationships in *general* and to combine *any* conception of didactics of mathematics with *any* conception of mathematics, with *any* kind of teaching practice, etc. I see no other way out of this situation than to start from some basic educational view on mathematics teaching and to select those attitudes towards mathematics, psychology, pedagogy and teaching practice which are compatible with this fundamental view. I am well aware that this approach explicitly involves subjectivism. However, subjectivism is already implicitly present and, moreover, there is always room for intersubjective agreement anyway.

The fundamental idea of mathematics teaching which is at the heart of the proposed approach—my ideology so to speak—is the following one:

> Mathematics teaching is doing mathematics with students in order to cultivate their understanding of reality.

Of course this "axiom" is nothing but a short formulation of a "genetic" view on mathematics teaching which has been developed elsewhere in detail (cf. Wittmann 1980b).

What are the consequences of this position for the fields surrounding the didactics of mathematics?

As far as mathematics is concerned, the genetic view of mathematics teaching emphasizes the dynamics of mathematics, its applications in small and big problems, the processes of solving problems, the relationships within mathematics, as well as to the outside world. Clearly this view is in sympathy with the lively picture of mathematics as described in Lakatos (1976), not with the anaemic skeleton usually presented to freshmen students.

Concerning psychology, those theories are of particular interest which are based on the learner's active search and which consider the learner's prerequisite knowledge as a crucial factor in the learning process. A typical example of this is J. Piaget's genetic epistemology and psychology.

Finally, within pedagogy both theories and methods of social learning have to be emphasized.

I would like at least to indicate the consequences to be inferred from these evaluations for the training of mathematics teachers. In order to be able to do mathematics with pupils the student-teacher needs:

(1) sufficient mathematical training in order to do mathematics at an appropriate level above the school curriculum;
(2) psychological training which introduces him or her to observing, analysing and understanding the successful and non-successful mathematical thinking processes of students;
(3) pedagogical training which incorporates understanding for social learning.

It is obvious that to achieve these goals we need teacher training programs quite different from most present ones. However, I cannot go into details here.

4 Teaching Units as the Integrating Core of Mathematics Education

The central thesis of this paper is this:

> In order to establish relationships between the different aspects of mathematics education and between the corresponding components of teacher training as well, it is useful to start from entities which already represent integration in a natural way, namely teaching units. Appropriate teaching units provide opportunities for doing mathematics, for studying one's own learning processes and those of students, for evaluating different forms of social organization, and for planning, performing and analysing practical teaching. Therefore teaching units are a unique means for penetrating all components of teacher training and relating them to one another.
>
> Finally, teaching units offer an excellent way for applied research into mathematics teaching.

I would like to illustrate this "philosophy of teaching units" by means of some special units.

4.1 Some Teaching Units

In my teacher training courses I regularly present a teaching unit (TU) in a format which involves brief information on: *objectives* (O), *materials* (M), mathematical *problems* arising from the context of the unit (P) and the—mostly mathematical, sometimes psychological—*background* of the unit (B). During the course these components are explained as much as I judge necessary.

The first three of the following examples are taken from the didactics of the primary level, the fourth from the secondary level.

4.1.1

TU	*Arithmogons* (McIntosh and Quadling 1975; Walther 1978)
O:	Adding, subtracting, operative investigation of these operations, searching-discovering.
M:	Trigonal and quadrilateral arithmogons (partly on worksheets).
P:	Given numbers in some vertices and edges. Find the other numbers!
B:	Linear independence of the numbers in vertices and edges, systematic solution by means of systems of linear equations, operative principle.[1]

As a reminder, arithmogons are triangles, quadrilaterals, in general n-gons, whose sides and vertices can be labelled by numbers according to the following rule: Each number in a side is the sum of the numbers in the adjacent vertices (cf. Fig. 2)

I'll come back to the mathematical background later.

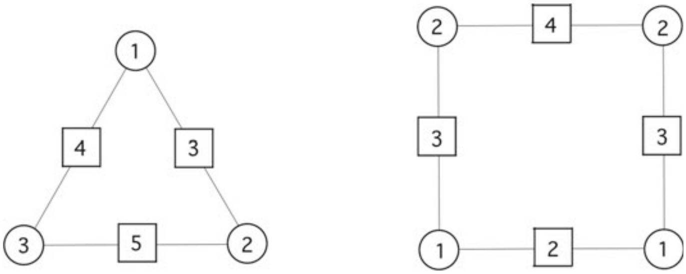

Fig. 2 Arithmogons: examples

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[1] The operative principle is explained in Wittmann (1980b), Chap. 8.

4.1.2

TU “*Chinese Remainder Theorem*”
O: Dividing with remainder, discovering, explaining.
M: Number line (integers).
P: Find a number which when divided by 3 leaves the remainder 1 and when divided by 4 leaves the remainder 2.
Find another number of this kind.
Find another one, . . . etc.
Can you find a pattern?
B: Chinese Remainder Theorem.

4.1.3

TU *Mini-Group-Ticket* (Wittmann 1980a)
O: Combinatorial counting, adding, subtracting, halving of amounts of money, mathematizing real situations, interpreting texts, reading tables.
M: Folder of the German Railway.
P: What is a mini-group ticket?
What is a mini-group?
How many mini-groups exist?
How much can you save by using a mini-group ticket?
B: Combinatorial counting, computational algorithms for addition and subtraction, functions.

4.1.4

TU *Galton Board* (Schupp 1976)
O: Mathematizing a stochastic situation.
M: Galton boards of various sizes, balls.
P: Where will the first ball fall?
Where will the second one fall?, etc.
Why?
What path can a ball take?
How many paths exist?
Which paths lead to the same goal?
Compare the probabilities of the paths, etc.
B: Bernoulli-chain, binomial distribution.

The examples are intended to show that a teaching unit in the sense of the present paper is not yet an elaborated plan for a series of lessons, although each one contains essential points of such a plan. Rather a teaching unit is an idea or a suggestion for a teaching approach which intentionally leaves various options of realisation open.

4.2 Teaching Units in Teacher Training

First I would like to show how teaching units can be used within different components of teacher training.

In *didactical* training proper, teaching units serve as illustrations of didactical conceptions of teaching certain mathematical ideas or concepts. In this sense Gerhard Müller and myself wrote a book for primary student teachers (Müller and Wittmann 1978), half of which consists of 24 carefully selected teaching units. These units represent the essential content, objectives, and materials of mathematics teaching at the primary level. On the other hand teaching units are useful references in courses on general didactical principles. Unit 4.1.1, for example, involves an application of the operative principle; unit 4.1.3 is a model for the genetic principle (see Wittmann 1980b, Chap. 10). My own experiences in teacher training have convinced me that *lectures* on basic issues of mathematics teaching are hard to bear, both for the professor and for the students, without reference to concrete teaching units.

In other words, the value of teaching units for didactical training is based on the fact that they organise didactical knowledge in an effective way for mathematics teaching.

As for *mathematical* training, arithmogons, for example, fit perfectly into a course on algebra for primary and secondary students. Students may first investigate arithmogons on their own and develop their own strategies for solving them.

There are different types of tasks.

(a) The numbers in the circles are given.
(b) Some numbers in the circles and some numbers in the squares are given.
(c) Only the numbers in the squares are given.

While in cases (a) and (b) the missing numbers can easily be determined by addition and subtraction, in case (c) problem solving strategies are needed. It turns out that trigonal arithmogons have always exactly one solution, while quadrilateral arithmogons have either no or more than one solution.

These experiences pave the way for a systematic algebraic treatment of arithmogons (McIntosh and Quadling 1975): The numbers $x_1, x_2, \ldots, x_n$ in the vertices and the numbers $y_1, y_2, \ldots, y_n$ in the edges of an n-gon (cf. Fig. 3) are related to one another by the linear equations

$$x_1 + x_2 = y_1, \quad x_2 + x_3 = y_2, \ldots, x_n + x_1 = y_n.$$

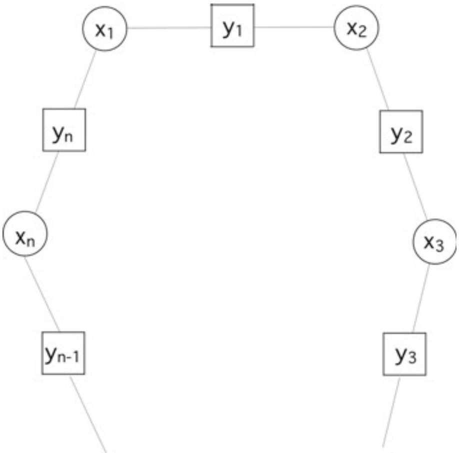

Fig. 3 Arithmogons: general setting

In other words, the mapping φ assigning the n-tuple $(x_1, \dots, x_n)$ to the n-tuple $(y_1, \dots, y_n)$ is a linear mapping of $\mathbb{Q}^n$ (or $\mathbb{R}^n$) into itself. Transforming the corresponding matrix shows that φ is of rank n, if n is odd and of rank $n - 1$, if n is even.

In the same manner the other units give rise to genuine mathematical work, as is indicated by the cues describing the mathematical background and need not be elaborated in detail here.

On the whole, I hope to have made clear that teaching units may serve as starting points for substantial mathematical activities on which important mathematical theories can be built. These theories are then obviously related to mathematics teaching. So prospective teachers will not think them irrelevant or useless, as is so often the case in today's teacher training. In my opinion a change of mathematical training towards the direction described here is one of the most urgent tasks in reforming teacher education.

As far as the *practical* training is concerned, it almost goes without saying that students should elaborate and test teaching units which they have met during their didactical training. In my opinion the crucial point for planning a lesson is to make the content or skills to be taught accessible by means of appropriate *problems* (cf. Wittmann 1980b, Chap. 5). This is the reason why "problems" is one of the four items to be considered in a teaching unit.

Videotaped units are very useful for illustrating didactical ideas and for stimulating students. In Dortmund we are going to establish a collection of videotapes of our favourite teaching units.

Because it is very difficult to study thinking processes of individual children in the classroom, student teachers should be offered *psychological-didactical studies* which introduce them to clinical interviews (cf. Herscovics and Bergeron 1980). My own approach to this field (Wittmann 1982) does not primarily use themes from the psychological literature, but rather, as might be expected, psychological problems arising from teaching units. For example, unit 4.1.3 poses the questions 'Which strategies do primary children use to find mini-groups?' and 'How many do they discover?' (cf. Heßler et al. 1980).

Last but not least, I should explain how to rethink the *pedagogical* training from the point of view of teaching units. In principle, teaching units are open to pedagogical considerations, too. I would like to refer here to teaching units which were especially devoted to organizing social activities (Müller and Wittmann 1978, Chap. 1.5.1, pp. 116 ff., Chap. 1.5.4, pp. 128 ff.; Wittmann 1977).

As mathematics educators are usually not as responsible for the pedagogical training as they are for the other components of teacher training, they cannot exert too much influence in this direction. It is important that pedagogues pick up teaching units and deepen them pedagogically.

4.3 Teaching Units in Didactical Research

Empirical research in mathematics education at the international level seems more and more in favour of clinical studies (Easley 1977). As long as such studies are done outside the natural context of learning and teaching and as long as they are directed towards basic research, however, they do not provide us with the knowledge necessary for guiding the learning of specific contents and procedures. For many years H. Freudenthal has developed a conception of mathematics education which is centered around the study of open and guided mathematical learning processes (cf. Freudenthal 1978). I would like to go a little step further and suggest that researchers investigate the mathematical thinking of pupils by using the framework provided by a series of didactically rich and widely accepted teaching units. The studies of Schupp (1976) with respect to unit 4.1.4 as well as those of Bell (1976), Galbraith (1981), may serve as models. Research of this kind would be immediately applicable to mathematics teaching by its very design. It would always include certain content and thus keep us away from unjustified generalisations over other contents.

The methodical degrees of freedom, offered when turning a variable teaching unit into a definite one, should be used by researchers in systematic variations of the conditions.

I would like to embed the "philosophy of teaching units" described in this paper into the methodology of the "Sciences of the artificial" created by the American Nobel Prize Winner Herb Simon (Simon 1970). Teaching units are just artificial objects constructed by mathematics educators, and it is my proposal to investigate the behavior and the adaptability of these objects to different educational ecologies.

5 Conclusion

The English group theorist Graham Higman once remarked "that progress in group theory depends primarily on an intimate knowledge of a large number of special groups". I believe that mathematics education could equally take substantial profit from the intimate knowledge of a large number of special teaching units.

This is not to question the importance of more general aspects of mathematics teaching beyond or independent of particular teaching units. Quite to the contrary: Just as group theory does not consist of a list of special groups, but is a theory surrounding and surmounting special groups, didactics of mathematics can only be viewed as a theory of and beyond real teaching.

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Chapter 3
Clinical Interviews Embedded in the "Philosophy of Teaching Units"—A Means of Developing Teachers' Attitudes and Skills

> *Those who like practice without theory are like seamen sailing without steering wheel and compass and never being sure where the voyage goes. Practice must always rest on good theory.*
>
> *Leonardo da Vinci*

Within the field of mathematics education there is at present a clearly increased interest in methodological issues. This fact cannot be explained only by internal motives but is also a reaction to pressures coming from well-established disciplines, which question the academic status of mathematics education, as well as from teachers' associations, which question the utility of mathematics education for practice. It is the relationship between theory and practice that lies at the very heart of the problem, and it is to be expected that first and foremost the elaboration of effective ways to relate theory and practice to each other will help to define the specific status of mathematics education, to prove its necessity, and thus to stabilize it, both internally and externally.

The theory-practice relationship is a problem that appears in more or less all applied fields of knowledge. So mathematics education can learn from the experiences of more advanced disciplines that support the following facts:

(1) The delineation of theory from practice is a natural and necessary step of development in any applied field and, in principle, opens the way to a more effective practice.
(2) The relationship between theory and practice cannot be fixed once and for all but must be continuously re-thought throughout the progressive development of the discipline in question.
(3) Tensions between theory and practice are not bad as such, but can be used for mutual criticism and thus as a source of progress.

Slightly revised version of a paper that appeared in: Christiansen, B. (ed.) (1984): Systematic Co-operation between Theory and Practice. Mini-Conference at ICME 5, Adelaide 1984. Copenhagen: Royal Danish School of Educational Studies. Dept. of Mathematics, 18–31.

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(4) There is always the tendency that theories will fix their own ends and develop independently of practice. This is no danger as long as they are related to a *significant core* which itself is related to practice in a vital way. If, however, theories develop in complete isolation from practice, they are bound to become useless and to degenerate.

At the present stage of mathematics education, the creation of such a *significant core* must be the primordial aim. I agree completely with the view expressed by Alan Bell (1985, 109):

> One might ask the general question whether, in the present state of knowledge about mathematical education, we should progress faster by collecting "hard" data on small questions, or "soft" data on major questions. It seems to me that only results related to fairly important practitioner questions are likely to become part of an intelligible scheme of knowledge. The developing theory of mathematical learning and teaching must be a refinement, an extension and a deepening of practitioner knowledge, not a separate growth. Specific results unrelated to major themes do not become part of communal knowledge. On the other hand "soft" results on major themes if they seem interesting and provocative to practitioners, get tested in the myriad of tiny experiments which teachers perform every day when they "try something and see if it works."

The present paper is based on an approach that tries to bridge the gap between theory and practice by means of teaching units (Wittmann 1984). While this approach is addressed to didactics as a whole, the perspective taken in this paper is that of pre-service teacher education. In the first part of the paper, John Dewey's position on the relationship between theory and practice is reviewed. The second and essential part will show how clinical interviews can be used to develop teachers' attitudes and skills within the "philosophy of teaching units." The paper is based on the booklet Wittmann (1982), which is a kind of reflection on the sabbatical leave spent by the author in Switzerland in 1974 where he had a chance to collaborate with Jean Piaget on the clarification of the concept of "grouping" (Wittmann 1978).

1 Cooperation Between Theory and Practice Through "Intermediate Practice"

From the viewpoint of the practitioner, the theory-practice interaction can be described by the scheme in Fig. 1 that is adapted from the well-known theory-practice loop in science. It reads as follows:

Within his or her daily practice the teacher is continuously confronted with (more or less open) teaching situations. In order to manage a given situation he or she uses his or her theoretical repertoire, his or her experience and various means for developing a model which indicates what to look for, what to do, what to expect, and which also explains his or her observations, decisions and prognoses. The model is a model "in the making," that is, it is being revised during the teaching process as indicated by the arrows.

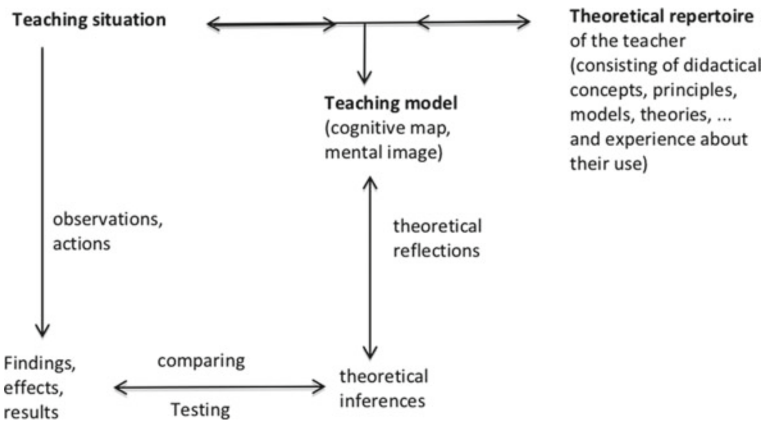

Fig. 1 The theory-practice loop of teaching

In the long run, experience with teaching models will lead to strengthening, weakening, modifying and revising the theoretical repertoire. So this repertoire itself is undergoing continuous development. By its very nature it may be called a *subjective theory* (or perhaps better, a collection of subjective theories) of the teacher. It must be distinguished from the theories of teaching developed within the discipline of mathematics education.

Using Karl Popper’s conception of the three worlds (Popper 1972), it can be stated: The teachers’ field of activity belongs to world 1, his subjective theory of teaching to world 2, whereas didactical theories of learning and teaching are part of world 3.

The central issue of teacher education is addressed by the following question: What is the best way to build up an effective theoretical repertoire for teaching?

One answer that has been given for centuries and is shared by the vast majority of practitioners even today is the *apprenticeship-conception of teacher education* (cf. Egsgard 1978): Suppose the prospective teacher knows his or her subject matter. Then the necessary theoretical tools evolve best through practice itself under the guidance of experienced teachers.

In a second view, the *scientific conception of teacher education*, which is held by most mathematics educators, the best professional preparation of teachers is seen in a study of mathematical, educational and didactical theories accompanied or followed up by practical work.

The two positions can only be evaluated and compared by referring to basic normative assumptions on mathematics teaching. The author of the present paper is in favor of a “genetic” perspective which can be characterized as follows:

(1) Mathematics is not just a collection of concepts, procedures and structures, but a living organism whose growth is stimulated by continuous attempts to solve big and small problems inside and outside of mathematics.

(2) Knowledge cannot be simply transmitted from the teacher to the learner, but must be developed (“constructed”) through the learner“s own activity.
(3) Social interaction is an essential component of learning and development.

Although the origins of the genetic view reach far back in history, this view has only received conscious attention since the beginning of the 20th century. Interestingly, a fundamental paper by John Dewey also elaborated on the relationship between theory and practice from this point of view at that early time (Dewey 1904/1977).

Dewey sees the essential task of a teacher in “directing the mental movement of students” and stimulating the “interaction of mind” (Dewey 1904/1977, 254):

> As every teacher knows, children have an inner and outer attention. The inner attention is the giving of the mind without reserve or qualification to the subject in hand. It is the first-hand and personal play of mental powers. As such it is a fundamental condition of mental growth. To be able to keep track of this mental play, to recognize the signs of its presence or absence, to know how it is initiated and maintained, how to test it by results attained, and to test apparent results by it, is the supreme mark and criterion of a teacher. It means insight into soul-action, ability to discriminate the genuine from the sham, and capacity to further one and discourage the other.

Dewey rejects the assumption held by the proponents of the apprenticeship-type of teacher education that the attitudes and skills of a good teacher can be acquired best through practice. On the contrary, he even considers premature practice as detrimental, because it puts the attention of the student teacher in the wrong place, and tends to fix it in the wrong direction, namely towards controlling the external attention of children, towards keeping them fixed upon his or her own questions, suggestions, instructions and remarks and upon their “lessons.”

According to Dewey, a reasonable practical training of student teachers is only possible (Dewey 1904/1977, 256)

> ... where the would-be teacher has become fairly saturated with his subject matter, and with his psychological and ethical philosophy of education. Only when such things have become incorporated in mental habit, have become part of the working tendencies of observation, insight, and reflection, will these principles work automatically, unconsciously, and hence promptly and effectively. And this means that practical work should be pursued primarily with reference to its reaction upon the professional pupil in making him a thoughtful and alert student of education, rather than to help him get immediate proficiency.

Dewey’s approach, which he calls the “*laboratory point of view,*” consists in forming the prospective teacher through “vital theoretical instruction.” Of course Dewey is far from understanding “vital theoretical instruction” as a transmission of mathematical, educational or didactical theories. His position is characterized by a very subtle analysis of what academic disciplines might contribute to a teacher’s subjective theory of teaching. What is important to him, as far as the subjects are concerned, is not the bulk of ready-made structures but the processes of thinking inherent in subject matter (Dewey 1904/1977, 263–264):

> There is therefore, method in subject matter itself – method indeed of the highest order which the human mind has yet evolved, scientific method. It cannot be too strongly emphasized that

> this scientific method is the method of mind itself ... [It] reflect[s] the attitudes and workings of mind in its endeavor to bring raw material of experience to a point where it at once satisfies and stimulates the needs of active thought. Such being the case, there is something wrong in the "academic" side of professional training, if by means of it the student does not constantly get object-lessons of the finest type in the kind of mental activity which characterizes mental growth and, hence, educative process.
>
> It is necessary to recognize the importance for the teacher's equipment of his own habituation to superior types of method of mental operation. The more a teacher in the future is likely to have to do with elementary teaching, the more, rather than the less, necessary is such exercise. Otherwise, the current traditions of elementary work with their tendency to talk and write down to the supposed intellectual level of children will be likely to continue. Only a teacher thoroughly trained in the higher levels of intellectual method and who thus has constantly in his own mind a sense of what adequate and genuine intellectual activity means, will be likely, indeed, not in mere word, to respect the mental integrity and force of children.

As far as teacher education is concerned, Dewey arrives at a surprising conclusion (Dewey 1904/1977, 260, 262):

> What the student [teacher] needs most at this stage of growth is ability to see what is going on in the minds of a group of persons who are in intellectual contact with one another. He needs to learn to observe psychologically – a very different thing from simply observing how a teacher gets "good results" in presenting any particular subject ... It is not too much to say that the most important thing for the teacher to consider, as regards his present relations to his pupils, is the attitudes and habits which his own modes of being, saying, and doing are fostering or discouraging in them. Now ... I think it will follow as a matter of course that only by beginning with the values and laws contained in the [student teacher's] own experience of his mental growth, and by proceeding gradually to facts connected with other persons of whom he can know little, and by proceeding still more gradually to the attempt actually to influence the mental operations of others, can educational theory be made most effective.

Dewey's position can be summarized with respect to mathematics teaching as follows: The main task of a teacher is to stimulate and to develop the mental activity and interaction of his or her pupils. The best way for a student teacher to acquire the necessary competence is to become familiar with mathematical thinking, to reflect upon these mathematical activities, to observe and analyze his or her own learning, in interaction with other student teachers, and to study the development of mathematical thinking in children and groups of children.

This kind of *doing mathematics* and *doing psychology* reflects the essential aspects of learning and teaching mathematics in the classroom. So it represents some sort of practice, which can be denoted as "*intermediate practice.*" The author of the present paper considers this kind of theory-based practice as the key to relating theory and practice in teacher education to each other.

From what has been said before it should go without saying that theoretical studies of mathematics, psychology and education in the sense of intermediate practice require an interdisciplinary approach and a re-organization of teacher education. This approach provides a real chance for mathematics education to fill a prominent place in teacher education programs.

2 Clinical Interviews as a Special Kind of Intermediate Practice

The importance of intermediate practice for prospective teachers would be greatly enhanced if doing mathematics and doing psychology could be linked to the school curriculum without, however, trivializing or distorting the requirements mentioned before. Wittmann (1984) suggested centering didactical research and development as well as teacher education around groups of teaching units which are sufficiently rich in order to allow for mathematical and psychological activities in the sense of intermediate practice and which are representative of the curriculum. This idea is not new. For example, it is developed to some extent in Fletcher (1965). Additional examples of this “philosophy of teaching units” for the primary level are provided by Müller and Wittmann (1984), Wittmann (1982). Clinical interviews fit in here in a quite natural way as will be shown in this section.

The clinical method and other protocol methods have been enjoying increased popularity as a research instrument among researchers in mathematics education (cf., for example, Easley 1977; Ginsburg 1983). Bergeron and Herscovics (1980) have also suggested using clinical interviews in teacher education.

At the university of Dortmund we started a “Development of Mathematical Thinking” course as part of our teacher education programs in 1975. In the first years this course was more or less an introduction into Piagetian psychology. Clinical interviews played an important but nevertheless subordinate role in the course. They were just used to illustrate Piaget’s theory. In this period the few interviews conducted by student teachers were just replications of experiments of the Genevan school.

Over the years the course has been revised considerably in several respects. As the pitfalls, inconsistencies and flaws of Piaget’s stage theory became more and more apparent (cf., for example, Brown and Desforges 1979; Groen and Kieran 1983) Piaget’s theory was pushed to the background in favor of his research method, the clinical interview.

As the time available for the course didn’t allow for a lengthy introduction into psychological theories anyway, we decided to concentrate the psychological-didactical training of our students on “*doing psychology*” in analogy to the emphasis we had put on “*doing mathematics*” within our mathematical courses. It seemed to us that clinical interviews are the easiest way of doing psychology.

Consequently, the course was split up into two parts:

(1) An introduction into basic ideas of Piaget’s genetic epistemology and into the clinical method,
(2) Clinical interviews with kindergarten or primary children conducted by the student teachers themselves.

The second part turned out to be extremely motivating for student teachers. It demonstrated that clinical interviews are a very valuable instrument for developing attitudes and skills of good teaching far beyond the psychological insights they may provide. Virtues of good teachers are: introducing children into a situation, making them feel comfortable, following their work and observing them without interrupting them, showing interest, listening carefully, accepting the children's thinking, avoiding criticism or authoritarian evaluation of the children's ideas, stimulating their thinking by cautiously arousing cognitive conflicts or by pointing to facts and statements that seem to have been overlooked, etc. (Wittmann 1982, preface).

A third modification of the course concerned the contents of the interviews. As already mentioned before, we started by replicating Genevan studies. However, the themes of these studies are far from the mathematics curriculum, in particular with respect to mathematical processes. So we replaced them more and more with themes taken from teaching units. The following examples are described through the suggestions and questions that "define" the interviews.

(1) Even and Odd Numbers

Questions:

- Can you tell me an even and an odd number? Is 5 even or odd? Is 10 even or odd? Why?
- Given a set of counters: How can you find out if the number of these counters is even or odd?
- The child is told the parity of each of two given sets of counters, each set with more than 10 counters, but the exact numbers are not given. The child is then asked to predict if the union of the two sets will be even or odd and to justify his or her answer.

(2) The Robbers and the Treasure

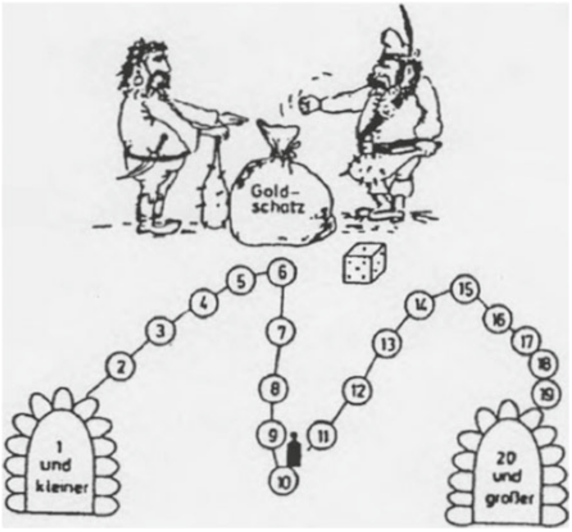

Fig. 2 Plan for playing "The robbers and the treasure"

The children are told a story (Müller and Wittmann 1984, 42): Two robbers are wrestling for a treasure. After some time there is no winner and they are exhausted. So they agree to resolve the quarrel by playing a game: They number a set of stones between their caves with numbers from 1 to 20 (Fig. 2). The treasure is put on field 10. Now they take turns throwing the die. According to the results, the treasure is moved towards the corresponding *cave*. As soon as the treasure enters a cave the owner of the cave wins it.

Suggestions and questions:

- Play the game with your partner.
- Suppose the treasure is on number 11: Where might it be after each of the two robbers has thrown the die just once?
- Where will the treasure be if the “plus robber” throws a “5” and the “minus robber” a “4?”

(3) Only Nine Digits

The child is provided with digit cards for the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9.

Suggestions and questions:

- Form two 3-digit numbers such that their sum is as big as possible.
- Why do you think your sum the maximum?
- Can you find other solutions?
- How can you make the sum as small as possible?

(4) A Word Problem

- In a class of 29 children there are three more girls than boys. How many girls and boys are there?

(5) The Ice Cream Problem

Suggestions and questions:

An ice cream seller offers four kinds of ice cream: chocolate, lemon, raspberry, pistachio. He sells cones with three scoops.

- How many different cones are possible?

(6) Northcott’s Nim

In this game of strategy, pairs of children are asked to play the game and to find out how to play as cleverly as possible (Wittmann 1982, 16–23).

In part, the themes for clinical interviews were inspired by theoretical considerations. We wanted to know how children of different ages would react to a given problem in order to use that knowledge for teaching. In many cases, however, the inspiration came from observing classes. One sees an interesting teaching episode, but the class moves on too quickly to the next activities. One wants to learn more about it and to understand the children’s thinking in more detail. For example, theme (3) was inspired by a lesson given by a primary teacher to third-graders.

For each year we have developed new sets of themes and offered them to pairs of student teachers for investigation. As a rule each pair selects a theme according to its taste and goes to a kindergarten or primary school, explains the tasks to the teachers and asks for collaboration. Supported by the teachers, the interviews are conducted with some 15 children, then transcribed and analyzed. Finally a report is written, presented and discussed in the seminar.

Some of the student teachers extend their clinical study into a thesis as part of their final examination. To give an example: One student teacher is presently working on theme (2) in cooperation with two schools. The teachers of grade 1 have identified children who have difficulties in adding and subtracting numbers. The student teacher uses the game both as a diagnostic and a remedial instrument. An interesting theoretical question here is the transition from material-based to mental calculations. At the same time, the study will provide information about the use of the game as a context for practicing computational skills. This kind of cooperation with schools looks quite promising.

Our experiences with the course in its new format are positive in two respects. First, the course fully serves its purpose as a framework for intensive intermediate practice. Clinical interviews with individual children or small groups of children in kindergarten or primary school represent a protected atmosphere where student teachers can concentrate on “intellectual contact,” “interaction of mind” and “mental movement,” to use Dewey’s terms. The student teachers are also stimulated to reflect on their own behavior and its influence on children. With some student teachers this results in quite a dramatic change of awareness. Later in their practical phase of teacher education (which in Germany follows university education and lasts two years), student teachers who reflect on their university studies retrospectively rate the relevance of the course very highly. The course ranks far ahead of all other courses. In particular, the student teachers appreciate the close connection to actual teaching practice.

Our second experience is that the skills of student teachers in conducting clinical interviews are a very good indicator of their skills in teaching a class. This is not surprising. As has already been mentioned before, the attitudes and skills of good teaching coincide with good attitudes and skills in conducting clinical interviews. So the course is very useful with respect to the personal development of student teachers as prospective teachers.

3 Concluding Remarks

Mathematics educators, who are in favor of major themes, basic ideas, and great lines of mathematics education, might find the occupation with teaching units as suggested in this paper to be conceptually poor, not controlled enough, not “research-oriented,” and perhaps naive. However, there are substantial arguments in favor of the “philosophy of teaching units” as a basis for the discipline of mathematics education.

1. Teaching units are a natural way to provide teachers and student teachers with a holistic view of mathematical, psychological, pedagogical, and practical aspects of mathematics teaching. This view is a specific mark of mathematics education.

2. Teaching units must not be seen in isolation, but rather in their relationship with objectives, contents and principles of mathematics teaching which they fill with meaning. Themes (2) and (3) of Sect. 2 are examples of a new approach to practicing skills (cf., for example, Winter 1984; Wittmann 1984). Themes (1) and (6) are typical for studying and developing mathematical processes (cf. Bell 1979). Theme (5) belongs to a class of units that are important for the development of combinatorial thinking, and theme (4) fits into the research on word problems. All themes could also be used to study social interaction in the classroom.

3. For bridging the gap between theory and practice it is necessary to have realistic empirical tests of theoretical ideas. It is only natural to use teaching units for infusing theoretical ideas with "meaning." Clinical interviews attached to teaching units offer excellent opportunities for intermediate practice and thus for shaping effective subjective theories of teaching (see Sect. 1).

4. The ability to do mathematics and to do psychology seems to be an essential prerequisite for making use of didactical theory in an intelligent way. Almost everything depends on self-reliant teachers equipped with heuristic strategies for selecting, modifying, rearranging, specializing, transferring, supplementing, and making practical what is offered to them. In order to be able to apply results of research in effective ways, teachers must to some extent be able to do research themselves. Preparing and conducting clinical interviews on fresh themes seems to be a good introduction.

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Chapter 4
The Mathematical Training of Teachers from the Point of View of Education

> *Incidentally, I dislike everything that is mere knowledge to me without extending my activity or directly invigorating me.*
>
> *J. W. von Goethe (in a letter to Friedrich Schiller)*

Summary

The paper describes an approach to integrating the mathematical and educational components in teacher training which is based on elaborating educational and psychological aspects *inherent in* “good mathematics”. This leads to a conception of informal, problem- and process-oriented presentations of elementary mathematics. The paper concludes by sketching an “elementary mathematics research program” in mathematics education.

Introduction

Since the middle of the seventies there has been a growing international discussion about what professional qualifications mathematics teachers need and what kind of training is appropriate to develop these qualifications (Bromme et al. 1981; Fletcher 1975; Otte 1979; Proceedings of ICME-4, Berkeley 1980, Chap. 5; Proceedings of ICME-5, Adelaide, 1984, Theme Group 3, pp. 146–158). It is not by chance that this discussion emerged in the seventies because, at that time, societal change in many countries around the world brought discord between two quite different philosophies of teacher training: that of training teachers for the grammar school, the college, the gymnasium, the lyceum, etc. and that of training teachers for the elementary school. The first of these two philosophies put prominent emphasis on subject matter as if familiarity with subject matter per se would form the only true scientific basis of teaching; vice versa, the other concentrated on pedagogy, psychology and methods courses and considered subject matter as a more-or-less trivial aspect of teaching (cf., Heintel 1978, pp. 12–22; Krämer 1987).

The recognition that the professional knowledge of all teachers has to be some synthesis of subject matter and educational knowledge helped to transform the historic controversy between different groups of teachers and teacher training philoso-

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phies into a structural problem of teacher training in general. As Otte (Otte 1979, pp. 114–115) comments:

> Compared with other professions, the special structural problem of the teaching profession is that it does not have one "basic science" such as law for the lawyer, medicine for the physician... Scientific theory is related in two utterly different ways to the practical work of mathematics teachers: first, scientific knowledge and methods are the subject matter of teaching; second, the conditions and forms of its transmission must be scientifically founded. Thus, teaching is under far more complex pressure than other professions to justify itself against competing conceptions of scientific theories, and has to cope with far greater demands with respect to the integration of diverse dimensions in the unity of action.

Similarly it was not by chance that in the seventies mathematics education (didactics of mathematics) emerged around the world as a specific interdisciplinary field of study related both to mathematics and to the educational sciences since a specific response to this structural problem of integrating the mathematical and educational aspects of teaching had to be developed which neither mathematics nor the educational sciences could provide.

Because of the inherent complexity of issues and because of historic burdens smooth and quick solutions of this problems of integration resolving all tensions between the different aspects are not likely, and so integration should be seen as an *extended* process which requires both time and deliberation. It is therefore appropriate to pause from time to time and reflect on what progress has been made, what deficiencies have been observed and to consider how to proceed further. The present paper is intended as a contribution to such an evaluation.

The structure of the paper is as follows:

I will first try to establish that a genuine integration of mathematics into mathematics education and a conception of embedding mathematics courses into a truly professional teacher training program are still lacking. Following John Dewey I will next show that a genuine integration of mathematics and education can be achieved only if educational and psychological relationships and processes *inherent in* good mathematics are elaborated and developed. This then leads me to a conception of informal, problem- and process-oriented presentations of elementary mathematics and their role in mathematics teacher training programs. I will conclude by identifying a research program on elementary mathematics.

1 The Problem of Integrating Mathematical and Educational aspects in Mathematics Education and Teacher Training

The diagram in Fig. 1 has been introduced to describe the locus of mathematics education and its interrelationships with the most important fields of reference (Wittmann Wittmann 1981, p. 2). Using this diagram I would like to describe the present state of integrating mathematical, educational and practical components into mathematics education as follows:

We have "flows" of theories from mathematics, psychology and pedagogy into mathematics education. Some of these are applied to didactical problems in an eclec-

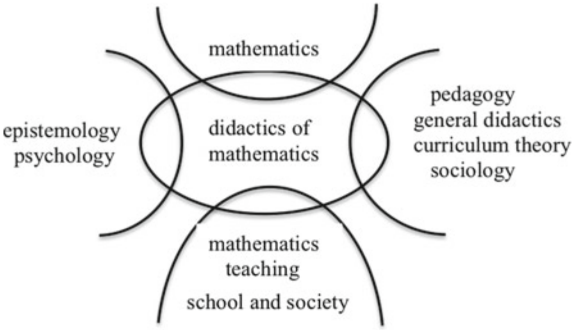


Fig. 1 Didactics of mathematics and the surrounding disciplines

tic way without, however, achieving a genuine integration. Although we can appreciate the contributions of mathematics educators to psychology I would nevertheless argue that presently research in mathematics education tends more to consuming "theories from abroad" than to producing its own "home-grown theories", to projecting its specific demands back to the related fields, and to working in these fields on its specific problems (Fig. 1).

We see a striking example of the lack of integration of mathematics into mathematics education in a recently published book by the international Bacomet-Group which set out "to consider, define, and analyse basic components of mathematics education for teachers" (cf. Christiansen et al. 1986). In a review of the book Quadling (1987, p. 188) makes the following statement:

> This is clearly an important book: the international perspective, the eminence of the collaborators, the resources which have supported its production, the targeting on a teacher-educator readership all mark it out for special attention. And yet in some respects the outcome is a disappointment. However stimulating the preparatory conferences were for the contributors, to the outsider there are few signs of common purpose... There must be concern also that the subject "mathematics" is curiously relegated to the side-lines, with many issues raised by Dörfler and McLone [in their article on school mathematics] ignored in later chapters...

The situation is not very different as far as the level of teacher training is concerned. In general, the mathematical training of teachers is not systematically related to educational aspects. Very often we find a formal study of mathematics ignoring the requirements of school in, as I see it, a scandalous way (Cooney 1988; Romberg 1988). The problem is particularly serious when the mathematical, the didactical and the educational training of teachers are in the responsibility of different faculties. However, organisation alone cannot explain the lack of integration. Even at institutions where the mathematical training of teachers is in the hands of mathematics educators there is often the same formal mathematical training as is typically found in departments of mathematics. This is mainly due to the fact that many mathematics educators tend to stick to their own scientific background either in mathematics or in the educational sciences and do not seriously strive for an *integration* of the two worlds.

2 The Educational Substance of Subject Matter

At a conference on "Trends and problems of mathematics teacher training" held in Bielefeld, FRG, Fletcher (1975, p. 217) asked the question "Is a mathematics teacher a mathematician or not?" and came to the following conclusion:

> The teacher of mathematics certainly needs to be a mathematician, and he needs to be a special sort of mathematician. He needs the general mathematical background that enables him to talk on equal terms with mathematics graduates, although he does not need some of the more specialised areas of mathematics which form part of most degree courses which are devoted exclusively to the subject. He needs a broad knowledge of applications in the world outside and in other parts of the school curriculum.
>
> In addition the teacher needs specialist skills of his own, in the translation of mathematics from one form into another, in understanding the pattern of thinking of his pupils at various stages of development and in understanding the relevance of structural ideas in mathematics to the teaching of it. Mathematics has its own criteria of truth, and the teacher has a special relation to his profession; if the teacher does not teach from conviction he alters the nature of the teaching he gives. The mathematics teacher is not only a mathematician, he is a professional mathematician with unique responsibilities.

A biased reader might perhaps understand Fletcher's paper as a plea for the traditional subject matter philosophy of teacher training referred to at the beginning of this paper, and consequently either support or reject it emphatically. However, a careful analysis of his paper shows that Fletcher is far from neglecting the educational component in the professional knowledge of mathematics teachers. In fact the basic message of his paper is that the educational knowledge cannot be acquired in separation from mathematics. In his contribution to ICME 4 Fletcher (1983, p. 113) put it this way:

> When we are teaching mathematics to prospective secondary teachers teaching method is not a subject apart. In many training institutions these two components are treated separately, but good mathematics and good methods can be studied simultaneously to the benefit of both. Let us make this our major fundamental change.

In my view, exactly this is the track we should follow in order to bring about a genuine integration between the mathematical and the educational knowledge of mathematics teachers. This perspective is further developed by reference to papers of two scholars from education and philosophy.

John Dewey (1904) wrote a fundamentally important article on the relation of theory to practice in teacher training. At that time the problems of integrating subject-matter preparation with professional instruction and relating educational theory to student-teaching were relatively new. It proves testimony to Dewey's genius that he at once hit the crucial points. Part B, section II, of Dewey's paper is devoted to training in subject matter and its relationship to educational theory and to practice:

> I turn now to the side of subject matter, or scholarship, with the hope of showing that here too the material, when properly presented, is not so *merely* theoretical, remote from practical problems of teaching, as is sometimes supposed... Scholastic knowledge is sometimes regarded as if it were something quite irrelevant to method. When this attitude is even unconsciously assumed, method becomes an external attachment to knowledge of subject-matter.

> It has to be elaborated and acquired in relative independence from subject-matter, and *then* applied.
>
> Now the body of knowledge which constitutes the subject-matter of the student-teacher must, by the nature of the case, be organized subject-matter... it has been selected and arranged with reference to controlling intellectual principles. There is, therefore, method in subject-matter itself – method indeed of the highest order which the human mind has yet evolved, scientific method.
>
> It cannot be too strongly emphasized that this scientific method is the method of the mind itself. The classifications, interpretations, explanations, and generalizations which make subject-matter a branch of study do not lie externally in facts apart from mind... It is necessary to recognize the importance for the teacher's equipment of his own habituation to superior types of method of mental operation. The more a teacher in the future is likely to have to do with elementary teaching, the more, rather than the less, necessary is such exercise... Only a teacher thoroughly trained in the higher levels of intellectual method and who thus has constantly in his own mind a sense of what adequate and genuine intellectual activity means, will be likely, in deed, not in mere word, to respect the mental integrity and force of children...
>
> The present divorce between scholarship and method is as harmful upon one side as upon the other - as detrimental to the best interests of higher academic instruction as it is to the training of teachers. But the only way in which this divorce can be broken down is by so presenting all subject-matter, for whatever ultimate, practical, or professional purpose, that it shall be apprehended as an objective embodiment of methods of mind in its search for, and transactions with, the truth of things.

Of course Dewey's ideas must be seen in the context of his holistic views about the relationship of subject-matter to life and learning and about the relationship of theoretical insight to practical activity in general. Ignoring this context could again easily lead to the wrong conclusion that Dewey pleaded for trivializing educational knowledge and for restricting teacher training to subject-matter training, which is certainly the last thing he wanted. In fact, Dewey argued in favor of elaborating the educational substance of subject-matter beyond the limitations of narrow subject specialists, as we shall see in the next section.

To reflect this unified picture of mathematical and educational components, didactics of mathematics as a specific professional discipline cannot be organized on the scientific level as a field of study where mathematical, psychological, pedagogical and practical aspects of teaching are investigated separately. Rather it must be equally based on a genuine integration of mathematics and the educational sciences. In a paper delivered at the first Symposium on mathematics education held in Klagenfurt, Austria, the philosopher Peter Heintel (1978, pp. 45–46) elaborated this position very clearly:

> Didactics of subject-matter means didactics rooted in subject-matter, in knowledge itself. It means analysing the subject according to didactical moments inherent or "deep-frozen" in itself.
>
> We have to start from the fact that the knowledge inherited in the subject has been the result of learning processes of mankind and that something of this genesis is still existent. Also we have to acknowledge the fact, that all knowledge, each subject and each science represent conventional and authorized systems of language, which govern not only our relations to nature, but also our social relations...

> ...Therefore taking subject matter fundamentally into account in building didactical models means breaking up the narrow boundaries of special disciplines, reconstructing "deep-frozen" learning processes, and elaborating the social use of knowledge and also its limitations.

3 Elementary Mathematics in Teacher Training

It goes almost without saying that the conception described above *depends crucially on a proper understanding of mathematics as* the subject of mathematics teaching. "Mathematics" must not be seen within the narrow boundaries of a specialised discipline which is represented exclusively by the departments of pure mathematics at the universities; rather it should be seen in the full spectrum of its relationships to science, to technology, to the humanities, and to human life. We should be reminded here of Whiteheads's famous dictum that there is only one subject matter of education, namely "Life in all its manifestations". Therefore the anemic, sterile presentations of mathematics as a closed formal system still widely in vogue around the world, are inappropriate for educational purposes. A genuine integration of the mathematical and educational aspects compels mathematics educators to develop courses which introduce mathematics as an integral part of human culture and in which preservice teachers learn mathematics *as a language* and *like a language*, not as a tacit and lonely game with "glass pearls". As a consequence there courses must present mathematics in a "mixed form" or as "interpreted mathematics" (cf. Dörfler and McLone 1986, p. 60), and they must concentrate on the more elementary fields of mathematics because those have the richest cultural interrelationships and therefore the strongest educational impact.

The pragmatic proposal presented here is this:

Every teacher training program should involve courses in *elementary mathematics* which are designed with respect to didactical, pedagogical and psychological aims. These courses should cover the whole mathematical training for primary teachers, for teachers of the lower secondary level a major part of it. These courses in elementary mathematics may be characterized as follows:

(1) The courses should be *explicitly related to the content of school mathematics* and give a coherent treatment of relevant parts of elementary algebra (number theory and combinatorics), elementary geometry, calculus, and elementary stochastics. However, they should go well beyond school mathematics in both depth and breadth.
(2) The courses should be *rich in relationships* to history, culture and the real world and should involve applications to mathematical phenomena in the environment of school students.
(3) The courses should be organized in a *genetic* way, i.e. they should be problem- and process-oriented. Theory should be developed through problems from inside and outside of mathematics with heuristics included in a prominent manner.

(4) The style of these courses should be *informal* ("inhaltlich-anschaulich") and involve a variety of means of representation using, in particular, concrete materials, pictures, diagrams, etc. This can be done in a *sound* way while avoiding sloppy and incorrect presentations. Logic must not be suspended in elementary mathematics.
(5) The courses should allow for a *variety of teaching/learning formats*, e.g. investigation, exposition, reading, and cooperative learning which involves discussion and a growing student awareness of their own learning processes.
(6) The courses will deal implicitly with the teaching of school mathematics, but they *need not* do so *explicitly*. This may be reserved for special didactical courses which are closely related to the courses in elementary mathematics. Although there are interesting attempts in primary teacher training to integrate mathematics and didactics within one course (cf. Goffree 1982; Davis 1987). I am personally in favor of a split for a pragmatic reason given already in Dewey's paper: "... the mind of a [teacher] student cannot give equal attention to both at the same time."

Comparing these characteristics with recent research on the nature of professional knowledge and mathematical literacy of mathematics teachers (Steinbring 1988; Romberg 1988) there can be no doubt that courses in elementary mathematics form an indispensable part of the professional training of mathematics teachers. In my view they represent what Fletcher (1975, p. 206) has called the teacher's "special mathematical expertise".

There is insufficient space in this paper to describe a course in elementary mathematics along the lines detailed above, but I would like at least to provide a glimpse of what elementary mathematics in this sense should be like. Let me give two examples.

The first one is from elementary algebra. My colleague Gerhard Müller and I are presently developing a mathematics course on arithmetic for primary teacher students in which counters and arrays of dots are used as a basic means of representation. For example, we follow a nice idea of Winter (1983) in deriving the divisibility rules. Students are given a place value chart with boxes for ones, tenths, hundreds etc. and are asked to solve a series of problems, as in the following:

(a) Find numbers up to 1000 which can be represented by 1, 2, 3, 4, 5 or 6 counters as in Fig. 2.
(b) Determine the remainders of each of these numbers with respect to the divisor 9.

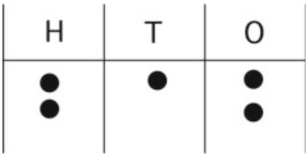


Fig. 2 Representing numbers by counters on the place value chart

The investigation shows that the remainders depend only on the number of counters, and the question arises why this is so. Answer: Moving a counter one box up or down changes the number by 9 (= 10 − 1) units and therefore the remainder does not change. This is an argument, both rigorous and general, which proves the divisibility rule for 9. By working with counters and searching for patterns all elementary divisibility rules can be discovered and proved in this manner.

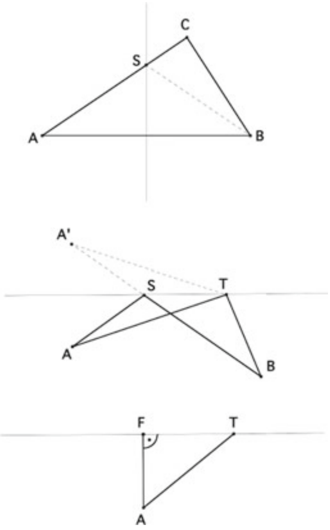
This little piece of substantial mathematics contains problems, theorems and sound proofs, and is clearly related to the teaching of arithmetic at the primary level. Further, it gives primary teacher students an important opportunity for investigation, provides them with a familiar means of representation (i.e., counters) which is fundamental for the primary level, and can serve as a model of how mathematics is developed. Since counters and the place value chart have also played an important role in history, this example meets all characteristics of the elementary mathematics courses listed above.

My second example is from elementary geometry. In my own course at the University of Dortmund I have included the following unit on mirrors (cf., Wittmann 1987, Chap. 1.1 and 3.2):

Starting from physical experiments with plane mirrors the concept of reflection is introduced and some inequalities are derived (Fig. 3).

The first two inequalities follow from the equal length of symmetric segments and the triangle inequality applied on the triangles BSC and BTA'.

Figure 3 (i) supports also the proof of a useful theorem that is an extension of the base angle theorem and its converse: The side opposite the bigger one of two angles of a triangle is longer than the side opposite the smaller angle and vice versa the angle opposite the longer one of two sides is bigger than the angle opposite the smaller side.



(i) Mirror inequality: $\overline{AC} > \overline{BC}$

(ii) Heron's inequality: $\overline{AT} + \overline{TB} > \overline{AS} + \overline{SB}$

(iii) Distance inequality: $\overline{AT} > \overline{AF}$

Fig. 3 Geometric inequalitites

The third inequality is a consequence of this theorem as the right angle in triangle is always the biggest angle.

These simple inequalities have surprisingly many applications.

Next the parabola, the ellipse and the hyperbola are defined by means of the well-known envelope constructions (Fig. 4).

Using the above inequalities tangents of these curves can be determined and then the focal properties are easy to prove. The transfer from the parabola to the ellipse and the hyperbola is a very nice exercise in heuristics. The unit ends with a study of technical applications of curved mirrors, e.g. telescopes, where combinations of different mirrors are used, and the kidney lithotripter developed a few years ago by the German company Dornier (Fig. 5).

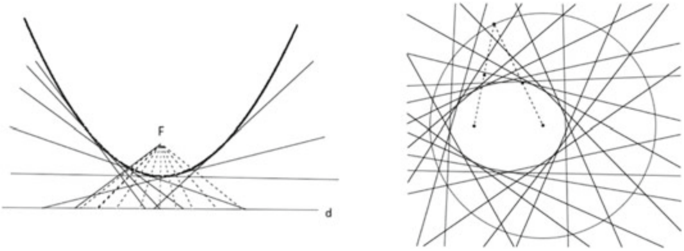
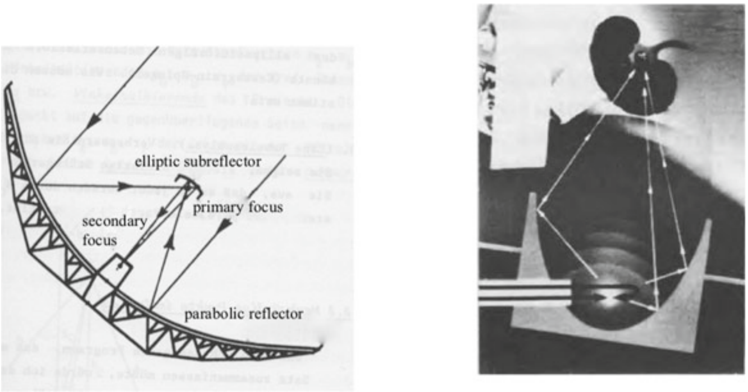


Fig. 4 Envelope constructions of the parabola and the ellipse



Gregory antenna

Extracorporeally induced destruction of kidney stones by means of shock waves

Fig. 5 Applications of elliptic and parabolic mirrors

Checking this unit against the items listed above shows that it meets all the required properties.

It can be seen that teaching proper didactical courses will be enhanced by courses in elementary mathematics if each one is tuned to the other; e.g., the divisibility rule for 9 could be re-met in a didactical course under the perspective of teaching it to children. Winter (1985) has reported on a teaching unit with 10 year-old-children which provides excellent material for didactical studies. Equally the example on mirrors can easily be related to the teaching of geometry at the secondary level. In this connection I would like to refer to the "philosophy of teaching units" (cf., Wittmann 1984; Becker 1986) which recommends the use of teaching units in a systematic way in order to relate different components of teacher training to one another.

4 The Elementary Mathematics Research Program of Mathematics Education

However reasonable and convincing the idea of integrating mathematical and educational aspects in teacher training may appear in principle, the difficulties of putting it into practice on a large scale should not be underestimated. In order to overcome them hard and extended research is needed.

Experiences in writing a textbook on elementary geometry for teacher training (cf., Wittmann 1987) have convinced me that the development of courses on elementary mathematics in the sense described above cannot be considered a more-or-less simple appendix to mathematics; quite on the contrary, *it presupposes the conceptual reconstruction of elementary parts of mathematics from an educational point of view*. Accordingly we have here a truly interdisciplinary task for which elements of mathematics, its history, its applications, aspects of epistemology, psychology, pedagogy and the mathematics curriculum have to be merged together. At first sight one might get the impression that this variety of aspects is an arbitrary mixture, but such is not the case. Behind the seeming diversity there is a common perspective—*the genetic principle of mathematics education* (cf., Wittmann 1981, Chap. 10) which unites all of the following:

(1) a genetic view on mathematics as expressed by Felix Klein and Henri Poincaré around the turn of this century and revived in our time by, for example, Freudenthal and Lakatos,
(2) Jean Piaget's genetic epistemology and Soviet psychology based on the concept of activity as the background of a great deal of work being done in the psychology of mathematics, and
(3) genetic theories of personal development and social interaction developed in both in the Western and Eastern World.

Although the literature of mathematics and its applications as well as the literature of mathematics education are full of beautiful examples of elementary mathematics which are consistent with the genetic view, nevertheless, *a coherent, homogeneous, and comprehensive conception of elementary mathematics is lacking* and therefore the position of elementary mathematics in teacher training is not as respected as it should be (cf. also, Fletcher 1975, p. 206). In my opinion this gap can only be filled if elementary mathematics is made a focus of didactical research. Therefore elementary mathematics is suggested as an important research context in mathematics education, with the vision to establish elementary mathematics as a kind of "natural mathematics", natural in the sense of "natural language" and "natural numbers".

In particular, this research should involve:

- the foundation of *informal mathematics as a self-consistent level of mathematical thought* beyond mathematical formalism (cf., Hanna 1983; Müller and Wittmann 1988),
- the development of *a "grammar" of non-symbolic means of representation* (cf., Sawyer 1964),
- the development of *operative proofs* (cf., Semadeni 1974; Kirsch 1979; Walther 1984),
- the development of *informal theories within contexts* (cf., "didactical phenomenology", Freudenthal 1983; Walther 1984; "interpreted mathematics", Dörfler and McLone 1986).

In this way elementary mathematics could become a substantial body of didactic knowledge with a unique profile which could compete in coherence and systematics with formal mathematics.

The pursuit of this research program will certainly not open a royal road to mathematics education but it will likely move us closer to an integration of mathematical and educational aspects within mathematics education and would greatly contribute to making our discipline an efficient professional background for mathematics teachers.

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Chapter 5
When Is a Proof a Proof?

In his famous talk at ICME 2 (Exeter 1972) the French mathematician R. Thom pointed out that any conception of mathematics teaching necessarily rests on a certain view of mathematics (Thom 1973, 204). As a consequence mathematics education cannot develop without close links to mathematics. However, "mathematics" must not be identified with the "official" picture of mathematics represented by lectures, mathematical journals and textbooks. What is needed is a fundamental and comprehensive view of mathematics as a cultural phenomenon including historic, sociological, philosophic and psychological aspects. Only this broader perspective permits us to recognize the "real" picture of mathematics and to use it for mathematics education.

In teacher training this extended perspective is necessary not only to provide prospective teachers in didactical courses with a sound meta-knowledge about mathematics but also with a productive relationship to school mathematics. During the past decade the investigations on the professional life of teachers have convincingly shown that school mathematics is not just a derivative of university mathematics but a relatively independent field, as it contains a variety of aspects which cannot be reduced to bare mathematical forms (Otte and Keitel 1979; Steinbring 1985, 1988; Dörfler and McLone 1986). As far as teacher training is concerned this new appreciation of school mathematics has influenced didactical and practical studies in pre-service and inservice training (Seeger and Steinbring 1986).

We believe, however, that in addition the mathematical training of student teachers has to be modified by including studies in elementary mathematics which emphasize meaning, process and informal means of representation, thus enabling student teachers to experience educational values of mathematics. Our position has developed through the past ten years while we have been strongly involved in reforming our teacher training programs. An example of what we have in mind is provided by a recent textbook on elementary geometry (Wittmann 1987).

The present paper aims at elaborating a central point of this kind of elementary mathematics, namely the notion of a sound informal proof. In German we use the term "inhaltlich-anschaulicher Beweis" in order to indicate that this method of demonstration calls upon the meaning of the terms employed, as distinct from abstract

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methods, which dispense with the interpretation of the terms and employ only the abstract relations between them.

During the seventies and eighties "proof" has been the subject of extended research in mathematics education (cf. the carefully collected list of references in Stein 1981, as well as Winter 1983a and Stein 1985). A completely new line of research has been opened up by Gila Hanna's book "Rigorous Proof in Mathematics Education" (Hanna 1983). This book was the first comprehensive attempt to transfer to mathematics education new tendencies in the philosophy of mathematics which originated in 1963 in Imre Lakatos' famous thesis "Proofs and Refutations" (Lakatos 1979).

The present paper is intended as a further contribution to the demystification of formalism in mathematics education. The first section shows by means of case studies that formalistic views on proof are still widely spread among mathematics teachers and student teachers. The second section gives some examples which show that in mathematical research informal and social aspects are highly important not only for finding but also for checking proofs. For overcoming these formalistic views we offer two strategies. One of them consists of referring to papers of leading mathematicians who give an authentic account of what their work is about. The other strategy is to elaborate informal mathematics as an independent level of mathematical thought. The third section of this paper explains in more detail what is intended by this "elementary mathematics research program of mathematics education".

1 Proofs and "Proofs"

The formalistic conception of mathematics, established during the first half of this century, defines mathematics as the science of "rigorous proof", i.e. the purely logical derivation of concepts from basic concepts and of theorems from axioms. As an illustration we refer to Pickert (1957, 49):

> Fortunately research into the foundations of mathematics – typical for this century – has developed a notion of mathematical proof which is independent of any imagination. I will start from this notion, explain it by means of a few examples and I will try to show by means of further examples what instruments are available in order to handle proofs in a more effective way, i.e. instruments which facilitate communication, retrieval and the discovery of proofs. It is the totality of these instruments which I would like to call "imagination" (Anschauung). In this way imagination is restricted to a certain domain: we use it as a guide, but we must not trust it. The validity of proofs depends only on what is left when imagination is completely removed. In my view this position is justified for the following reasons: first I do not see how generally accepted decisions on the validity of a proof could be made otherwise - i.e. by using imagination. Second I believe that this position reflects the view of contemporary mathematicians.

The following quotation from MacLane (1981, 465) is even more pointed:

> This use of deductive and axiomatic methods focuses attention on an extraordinary accomplishment of fundamental interest: the formulation of an exact notion of absolute rigor. Such a notion rests on an explicit formulation of the rules of logic and their consequential and

> meticulous use in deriving from the axioms at issue all subsequent properties, as strictly formulated in theorems. Moreover, each derivation can be tested and understood in its own terms, independent of any reference to examples of the activity or the reality for which the axioms were designed ... This formal character of mathematics may serve to distinguish it from all other types of science. Once the axioms and the rules are fully formulated, everything else is built up from them, without recourse to the outside world, or to Intuition, or to experiment. An absolutely rigorous proof is rarely given explicitly. Most verbal or written mathematical proofs are simply sketches which give enough detail to indicate how a full rigorous proof might be constructed. Such sketches thus serve to convey conviction – either the conviction that the result is correct or the conviction that a rigorous proof could be constructed. Because of the conviction that comes from sketchy proofs, many mathematicians think that mathematics does not need the notion of absolute rigor and that real understanding is not achieved by rigor.
>
> Nevertheless, I claim that the notion of absolute rigor is present.

While MacLane like Pickert refer to intuition at least as a means for handling and evaluating proofs, the logician Rosser (1953, 7) takes an extreme position:

> Thus, a person with simple arithmetical skills can check the proofs of the most difficult mathematical demonstrations, provided that the proofs are first expressed in symbolic logic. This is due to the fact that, in symbolic logic, demonstrations depend only on the forms of statements, and not at all on their meanings.
>
> This does not mean that it is now any easier to discover a proof for a difficult theorem. This still requires the same high order of mathematical talent as before. However, once the proof is discovered, and stated in symbolic logic, it can be checked by a moron.

What role formalism has played in relationship with or in opposition to other philosophic positions and how it has developed into the "official" philosophy of mathematics is explained by Davis and Hersh (1983, Chap. 7). In addition these authors describe how during the past ten years quite different views have gained ground in which the possibility of absolutely rigorous and eternal proofs has been denied and in which proof is considered as a social process among mathematicians:

> A proof only becomes a proof after the social act of "accepting it as a proof". This is as true for mathematics as it is for physics, linguistics, or biology. The evolution of commonly accepted criteria for an argument's being a proof is an almost untouched theme in the history of science. (Manin 1977, 48).

Going one step further leads to the possibility of different criteria for checking and evaluating proofs within different social contexts. An important example, applied mathematics, is analyzed by Blechman, Myschkis and Panovko (1984).

We will now show by means of four case studies, ranging from primary mathematics teaching to teacher training, how mathematical understanding can be inhibited by formalistic views on proof. Our experiences with student teachers are concentrated on the primary and the lower secondary level. However, our contacts with student teachers for the upper secondary level in didactical courses have convinced us that formalism is represented even more strongly in this group than in the two other groups—as one would expect.

Example 1 (*Chinese Remainder Theorem*) This theorem was part of a course in number theory for student primary teachers. Some students felt overwhelmed and protested at the inclusion of this—in their view useless—topic into primary teacher programs. In defence the students were told that the Chinese Remainder Theorem could suggest some interesting work with 9-year old children. In order to substantiate this claim it was agreed to pose the following problem to third graders:

> Find numbers which leave the remainder 1 when divided by 2, and the remainder 2 when divided by 3.

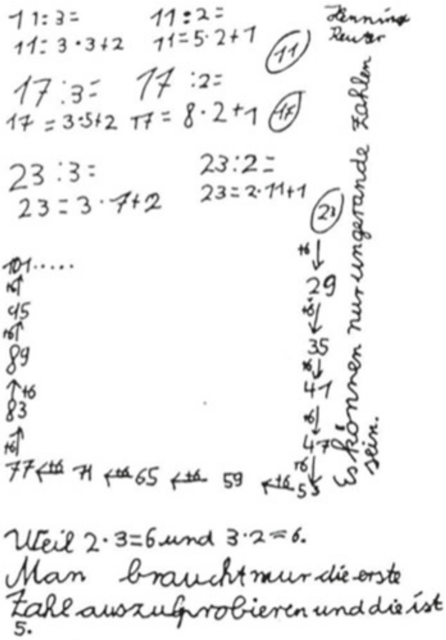
Of course this problem was not introduced in this compact form, but was explained by checking small numbers step by step. In particular 5 was stated as the smallest number with the required properties:

$$5 = 2 \cdot 2 + 1, \qquad 5 = 1 \cdot 3 + 2.$$

Afterwards the children started their own search. Children who had found a series of solutions were stimulated to find all solutions.

The spectrum of achievements was considerable. Some students still had problems with the calculations, others paid attention only to the remainder 2. The best performance was by Henning, according to the teacher the "mathematician" of the class. It is represented here in facsimile (Fig. 1).

Obviously Henning had checked number by number and had found the solutions 11, 17, 23. Then he recognized a pattern which he summarised by the statement:



Es können nur ungerade Zahlen sein.

Weil 2 · 3 = 6 und 3 · 2 = 6.
Man braucht nur die erste Zahl auszuprobieren und die ist 5.

Fig. 1 Student solution

"Because $2 \cdot 3 = 6$ and $3 \cdot 2 = 6$." When asked to explain this subtle argument Henning said something like this:

> If I consider the remainder 1, I have to proceed in steps of 2 and I meet the odd numbers. If I consider the remainder 2, I have to proceed in steps of 3. The steps coincide only after 3 two-steps and 2 three-steps.

In our view this argument is a sound informal demonstration of the solution. The student teachers, however, were not willing to accept it as a proof. In their view a "real" proof had to be based an formal transformations of a system of congruences—something going obviously beyond the capabilities of primary children. Therefore they continued to reject the Chinese Remainder Theorem as a topic appropriate for primary teacher training.

Example 2 (*Irrationality of* $\sqrt{2}$) Pickert (1987, 212) presents the following proof of the irrationality of $\sqrt{2}$ by a 13-year-old student:

> Let $a, b \in N^*$ be relatively prime such that $(a/b)^2 = 2$. Then $a^2 = 2b^2$ and therefore b^2 is a common divisor for a^2 and b^2. As a and b are relatively prime, so are a^2 and b^2. As a consequence $b^2 = 1$ and so $a^2 = 2$, which is impossible for $a \in N^*$.

According to Pickert this argument is a "proof", because the student uses tacitly the inference

$$a, b \ \text{ relatively prime } \implies a^2, b^2 \ \text{ relatively prime,}$$

which does not hold in rings in general. In our view this position is too formalistic. The inference used by the student is a direct consequence of the unique factorisation of natural numbers into prime numbers. The latter is well known to students in grade 7 because it has been treated in grade 5 and is used all the time for cancelling fractions. The student is right to use "socially shared" knowledge implicitly. The above mentioned inference becomes crucial only within ring theory—a context completely irrelevant at school.

Example 3 (*Euler's polyhedron theorem*) In a lecture "Geometry in 3-space" for primary students, Euler's theorem was proved for convex polyhedra in the following informal way:

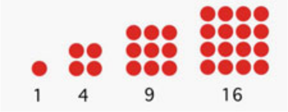
First the concept of a Schlegel diagram was explained and as an illustration, the Schlegel diagrams for some polyhedra were produced by means of rubber sheets. Then the relationship $v + f - e = 2$ was proved by showing that an arbitrary Schlegel diagram can be reconstructed by starting with one point ($v = 1$, $f = 1$, $e = 0$) and adding edge by edge such that $v + f - e$ does not change (cf. Wittmann 1987, 270ff.) Right after the demonstration a student asked "Was that really a proof?" The teacher, somewhat irritated by this unexpected question, asked back: "Why not?", and received a very instructive answer: "Because I understood it!"

A conversation later on showed clearly that the student had difficulties with the formalistic teaching received at school and had consistently arrived at the conclusion that mathematical proofs were not accessible to her.

In our view this student is not a single case but represents a great number of students.

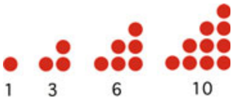
Example 4 (*Trapezoid numbers*) Experiences like that in example 3 have stimulated us to investigate the mathematical "world view" of our students more systematically. The method we have found most useful is to confront the students with informal and formal proofs and to ask them to evaluate the validity of each type.

In this way primary student teachers were introduced to the old Greek "arithmetic of dot patterns" (cf. Becker 1954, 34 ff.). For example starting with square numbers (Fig. 2) and the triangular numbers (Fig. 3) trapezoid numbers were defined as composition of square and triangular numbers (Fig. 4).



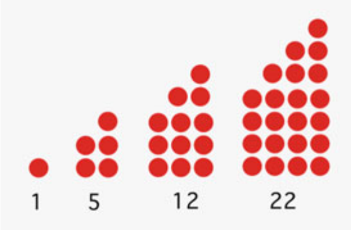
$$Q_n = n^2.$$

Fig. 2 Square numbers



$$D_n = n(n+1)/2.$$

Fig. 3 Triangular numbers



$$T_n = n^2 + (n-1)n/2 = (3n^2 - n)/2.$$

Fig. 4 Trapezoid numbers

By "playing" with patterns the students guessed that for all n the trapezoid number T_n and n leave the same remainder when divided by 3:

$$T_n \equiv n \pmod{3}.$$

The teacher offered the following "iconic" proof (Fig. 5):

Starting from the right the pattern T_n is decomposed into columns. Obviously each 3-column is a multiple of 3. If n itself is a multiple of 3 (case 1) then T_n splits completely into 3-columns and is also a multiple of 3. If n leaves the remainder 1 (case 2) then T_n splits into 3-columns and a single column with n points at the left side and it is again obvious that T_n leaves the remainder 1, too.

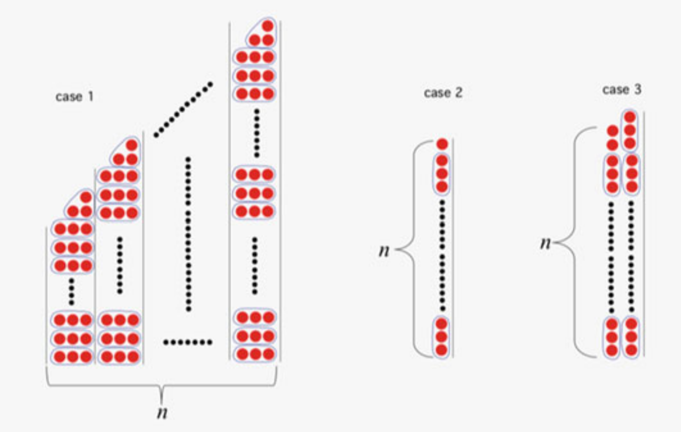


Fig. 5 Decomposition of trapezoid numbers modulo 3

Finally, if n leaves the remainder 2 (case 3), T_n breaks up into 3-columns and two columns with n and $(n + 1)$ points at the left side, and it is easy to see that T_n like n leaves the remainder 2.

Right after this demonstration some students expressed their doubt on the validity of it. The teacher didn't intervene and the group agreed very quickly that the demonstration could achieve only the status of an illustration, and not the status of a proof.

The teacher then offered the following "symbolic" proof:

Case 1: Let $n = 3k$. Then $T_n = (3n^2 - n)/2 = n(3n - 1)/2 = 3k(9k - 1)/2$, and as k or $9k - 1$ are even, T_n (like n) is divisible by 3.

Case 2: Let $n = 3k + 1$. Then $T_n = (3(9k^2 + 6k + 1) - (3k + 1))/2 = (27k^2 + 15k + 2)/2 = (27k^2 + 15k)/2 + 1 = 3k(9k + 5)/2 + 1$,
i.e. T_n (like n) leaves the remainder 1.

Case 3: Let $n = 3k + 2$. Then $T_n = 3(9k^2 + 12k + 4) - (3k + 2))/2 = (27k^2 + 33k + 10)/2 = 3(9k^2 + 11k + 2)/2 + 2$,
i.e. T_n (like n) leaves the remainder 2.

The confrontation of the iconic and the symbolic argument aroused a lively discussion on the validity of each, in which the teacher defended the iconic argument as a sound proof. The students were then asked to give written comparisons of the two forms of proof. The papers showed very clearly how strongly the teaching received at school had predisposed the students towards formal proofs and how difficult it was for them to accept an "iconic" proof. As an illustration we quote from some papers:

The iconic proof is much more intuitive for me and it explains to me much better what the problem is. For me dot patterns are convincing and sufficient as a proof. Unfortunately we have not been made familiar with this type of proof at school … Only symbolic proofs have been taught.

For me the iconic proof is easier and better to understand than the symbolic proof. The reason is that it is much more intuitive than the symbolic one. I have tried to follow the symbolic proof by verification. However, the calculation is somewhat abstract, and I cannot link anything concrete to it.

The symbolic proof is to be preferred, because it is more mathematical.

The iconic proof is highly intuitive. It shows very clearly that T_n is divisible by 3 if n is divisible by 3, and vice versa. It is true that the iconic proof does not allow to substitute arbitrarily large numbers for n and to prove the statement for them because so many dots cannot be drawn; but for smaller numbers the representation by means of 3-columns is very useful for the understanding.

The iconic proof is very intuitive. One understands the connection from which the statement flows. I can't imagine how a counterexample could be found, because it does not matter how many 3-columns are constructed. In my opinion this is nevertheless no proof, but only a demonstration which, however, holds for all n. At school I have learnt that only a symbolic proof is a proof. Therefore I trust such proofs more. As symbolic proofs are more or less just "calculating to and fro" one easily loses sight of what is to be proved. The confrontation of the two types of proof seems very instructive to me.

The iconic proof is much easier to understand and more intuitive than the symbolic one. At school there were mainly symbolic proofs. Iconic proofs were only means for finding symbolic proofs. I still have this feeling.

I prefer the symbolic proof, as school has confronted me only with this type of proof. These proofs guarantee generality. The iconic proof is more intuitive, which is surely an advantage for primary teaching. I for myself see the iconic proof more as an illustration and concretisation of the symbolic proof.

The symbolic proof is more mathematical. This proof is more demanding, as some formulae are involved which you have to know and to retrieve. The iconic proof can be followed step by step, and each is immediately clear. However, I wonder if an iconic proof would be accepted in examinations.

Personally I like the iconic proof as it is more intuitive. You can see at once what's going on whereas the symbolic proof forces you to think in an abstract way. You know the formula and you develop through abstract thinking (?) the proof, but you don't have a direct reference to numbers. I prefer the symbolic proof as I have been confronted with such proofs at school and as it is more mathematical.

I am familiar with the symbolic proof. Therefore it is easier for me to handle.

I prefer the iconic proof because of its intuitive character. To me the symbolic proof is too abstract. Possibly I could have discovered the iconic proof for myself. Nevertheless I always try to find a symbolic proof, presumably because of my former mathematics teaching.

Influenced by the mathematics teaching at school and at the university I would prefer the symbolic proof. However, the iconic proof is much more convincing, as it is less abstract and easier to verify. Up to now iconic proofs were unknown to me.

Normally I trust symbolic proofs more, as they use general "numbers" (variables), i.e. they cover any number in any case. However, in this example I trust the iconic proof, too. It is more intuitive and the idea of the proof is clear and obvious.

To me the iconic proof is mathematical enough – I do not mistrust it!! One important point for me is that children at the primary level can understand things better through intuition.

> Judging by the objectives of my study as a prospective primary teacher I cannot see very much reason for symbolic proofs – in particular when they become even “more mathematical”. I miss the practical impact.

What these statements show is reinforced by our experiences in teacher training: the vast majority of student teachers holds a definite formalistic view on mathematical proof (cf. also Aner et al. 1979). Obviously this fact is only a reflex of the “official” picture of mathematics prevailing at school and in teacher training for a very long time. Branford (1913, 328) has recognized this problem very clearly at the beginning of this century:

> I think it a fact that the vast majority of teachers is firmly convinced that mathematics does not differ so much from other sciences by the measure of rigour but by the absolute rigour of mathematical proofs in contrast to the approximate rigour of other proofs.

And Branford continues:

> The disaster caused by this belief at all levels of mathematics teaching is, I think, terrible.

The negative consequences of a formalistic understanding of proof by teachers can be quite different:

Teachers who see themselves as pedagogues, pragmatists and teachers with a negative attitude towards mathematics refrain from introducing their students to proof because they think “mathematical” proofs are too difficult for their children. Instead they offer pictures, plausible arguments, verifications, examples and rules related to certain types of tasks. Lenné (1969, 51) called this didactical position the “didactics of tasks” (Aufgabendidaktik). On the other side teachers who care for “mathematical rigour” try to bring their teaching to the level of formal definitions and proofs, even if in practice they do not get very far. Systematic attempts in this direction were made by the “New Maths” movement in the sixties and seventies. Here elements of formal university mathematics were more or less directly “mapped down” to the level of teaching. This approach has therefore rightly been called the “didactics of mapping down” (Andelfinger and Voigt 1986, 3).

Contrary to the “didactics of tasks” and the “didactics of mapping down” there is a third branch of didactics, centered around the genetic principle, in which the concept of informal proof has been developed. Branford, one of the most important representatives of genetic didactics distinguishes three types of proof (Branford 1913, 100 ff., 239 ff.):

- experimental “proofs” (typical of the “didactics of tasks”)
- intuitive proofs
- scientific proofs (typical of university mathematics and the “didactics of mapping down”)

Branford thinks that the middle type of proof, intuitive proof, is indispensable for the development of mathematical understanding and characterises it in the following way:

> This level of proof establishes general and rigorously valid truths. However it refers, if necessary, to postulates of sensual perception. It puts truth on a basis of its own by immediate recall to first principles. It does not represent truth as a mere link in a systematic chain of arguments where the effective strength of the connection is weakened by the number of previously stated truths forming the links of the chain (103) …
>
> Opposite to experimental "proof" we find the two other types of proof, the scientific and the intuitive proof, the latter being a more preliminary and less rigorous kind of the ideal scientific proof; in reality there is no sharp border between these two types, they differ only in the degree of logical rigour. The truths derived by each of the two types are generally valid as far as we can judge. Otherwise sensual perception would show us exceptions (108 f.)

With perfect clarity Branford points out here that the border between "proof" and proof does not lie between "experimental" and "intuitive" proof on the one side and purely logical proofs on the other side but is to be drawn between experimental proofs on the one side and the two other types on the other side. Later on this position was elaborated in more detail, in particular by H. Freudenthal's contributions (Freudenthal 1963, 1973, Chap. 8, 1979). The decisive distinction between experimental "proofs" and intuitive, informal proofs has been clarified by didactical research on operative proofs and on "pre-mathematics" (Semadeni 1974; Kirsch 1979; Winter 1983b; Walther 1984): Experimental "proofs" consist of the verification of a finite number of examples guaranteeing of course no generality. Informal operative proofs are based on constructions and operations which by intuition are seen as applicable to a whole class of examples and as leading to certain consequences. For example the decomposition of a trapezoid point pattern into 3-columns is a universal operation which generates insight into the remainder. The dot pattern is not a picture here but a symbol (cf. Jahnke 1984).

In spite of its advanced development the genetic position has not received much attention as yet in mathematics teaching at any level. The main reason might be that informal explanations of concepts and informal proofs seem inhomogeneous, unsystematic, shaky and invalid when considered from the point of view of formalism. Many teachers, textbook authors and teacher trainers refrain from representations which might be interpreted as a sign of mathematical incompetency. A change in this unfavourable situation can be expected only to the extent that formalism is overcome as the "official" philosophy of mathematics and that comprehensive informal conceptions of elementary theories of mathematics are developed. These two points are considered in the following sections.

2 Formalism as a Fiction: The Indispensability of Intuition and Social Agreement in Checking Proofs

As already mentioned in the introduction of this paper at present the philosophy of mathematics is undergoing a dramatic change arising from a growing awareness of working mathematicians that formalism is in contradiction to their experiences and that the ideal of an "absolutely rigorous" proof can no longer be maintained (Davis and Hersh 1983, Chap. 7; Hanna 1983).

We would like to illustrate the new views an proof by quoting from papers of leading mathematicians.

Hardy (1929, 18 f.):

> I have myself always thought of a mathematician as in the first instance an *observer*, a man who gazes at a distant range of mountains and notes down his observations. His object is simply to distinguish clearly and notify to others as many different peaks as he can. There are some peaks which he can distinguish easily, while others are less clear. He sees A sharply, while of B he can obtain only transitory glimpses. At last he makes out a ridge which leads from A, and following it to its end he discovers that it culminates in B. B is now fixed in his vision, and from this point he can proceed to further discoveries. In other cases perhaps he can distinguish a ridge which vanishes in the distance, and conjectures that it leads to a peak in the clouds or below the horizon. But when he sees a peak he believes that it is there simply because he sees it. If he wishes someone else to see it, he *points to it*, either directly or through the chain of summits which led him to recognise it himself. When his pupil also sees it, the research, the argument, the *proof* is finished.
>
> The analogy is a rough one, but I am sure it is not altogether misleading. If we were to push it to its extreme we should be led to a rather paradoxial conclusion; that there is, strictly, no such thing as mathematical proof; that we can, in the last analysis, do nothing but *point*; that proofs are what Littlewood and I call *gas*, rhetorical flourishes designed to affect psychology, pictures on the board in the lecture, devices to stimulate the imagination of pupils. This is plainly not the whole truth, but there is a good deal in it. The image gives us a genuine approximation to the processes of mathematical pedagogy on the one hand and of mathematical discovery on the other; it is only the very unsophisticated outsider who imagines that mathematicians make discoveries by turning the handle of some miraculous machine …
>
> On the other hand it is not disputed that mathematics is full of proofs, of undeniable interest and importance, whose purpose is not in the least to secure conviction. Our interest in these proofs depends on their formal and aesthetic properties … Here we are interested in the pattern of proof *only*. In our practice as mathematicians, of course, we cannot distinguish so sharply, and our proofs are neither the one thing nor the other, but a more or less rational compromise between the two. Our object is *both* to exhibit the pattern and to obtain assent. We cannot exhibit the pattern completely, since it is far too elaborate; and we cannot be content with mere assent from a hearer blind to its beauty.

Wilder (1944, 319):

> In conclusion, then, I wish to repeat my belief that what we call "proof" in mathematics is nothing but a testing of the products of our intuition. Obviously we don't possess, and probably will never possess, any standard of proof that is independent of time, the thing to be proved, or the person or school of thought using it. And under these conditions, the sensible thing to do seems to be to admit that there is no such thing, generally, as absolute truth in mathematics, whatever the public may think.

Thom (1973, 202 ff.):

> The real problem which confronts mathematics teaching is not that of rigour, but the problem of the development of "meaning", of the "existence" of mathematical objects.
>
> This leads me to deal with the old war-horse of the modernists (of the Continental European variety): rigour and axiomatics. One knows that any hope of giving mathematics a rigorously formal basis was irreparably shattered by Gödel's theorem. However, it does not seem as

> if mathematicians suffer greatly in their professional activities from this. Why? Because in practice, a mathematician's thought is never a formalised one. The mathematician gives a meaning to every proposition, one which allows him to forget the formal statement of this proposition within any existing formalised theory (the meaning confers on the proposition an ontological status independent of all formalisation). One can, I believe, affirm in all sincerity, that the only formal processes in mathematics are those of numerical and algebraic computation. Now can one reduce mathematics to calculation? Certainly not, for even in a situation which is entirely concerned with calculation, the step of the *calculation* must be chosen from a very large number of possibilities. And one's choice is guided only by the intuitive interpretation of the quantities involved. Thus the emphasis placed by modernists on axiomatics is not only a pedagogical aberration (which is obvious enough) but also a truly mathematical one.
>
> One has not, I believe, extracted from Hilbert's axiomatics the true lesson to be found there; it is this: one accedes to absolute rigour only by eliminating meaning; … But if one must choose between rigour and meaning, I shall unhesitatingly choose the latter. It is this choice one has always made in mathematics, where one works almost always in a semi-formalised situation, with a metalanguage which is ordinary speech, not formalised. And the whole profession is happy with this bastard situation and does not ask for anything better. …
>
> A proof of a theorem (T) is like a path which, setting out from propositions derived from the common stern (and thus intelligible to all), leads by successive steps to a psychological state of affairs in which (T) appears obvious. The rigour of the proof – in the usual, not the formalised, sense – depends on the fact that each of the steps is perfectly clear to every reader, taking into account the extensions of meaning already effected in the previous stages. In mathematics, if one rejects a proof, it is more often because it is incomprehensible than because it is false. Generally this happens because the author, blinded in some way by the vision of his discovery, has made unduly optimistic assumptions about shared backgrounds. A little later his colleagues will make explicit that which the author had expressed implicitly, and by filling in the gaps will make the proof complete.

Atiyah (1984, 16 ff.):

> If I'm interested in some topic then I just try to understand it; I just go on thinking about it and trying to dig down deeper and deeper. If I understand it, then I know what is right and what is not right.
>
> Of course it is also possible that your understanding has been faulty, and you thought you understood it but it turns out eventually that you were wrong. Broadly speaking, once you really feel that you understand something and you have enough experience with that type of question through lots of examples and through connections with other things, you get a feeling for what is going on and what ought to be right. And then the question is: How do you actually prove it? That may take a long time …
>
> I don't pay very much attention to the importance of proofs. I think it is more important to understand something …
>
> I think ideally as you are trying to communicate mathematics, you ought to be trying to communicate understanding. It is relatively easy to do this in conversation. When I collaborate with people, we exchange ideas at this level of understanding – we understand topics and we ding to our intuition.
>
> If I give talks, I try always to convey the essential ingredients of a topic. When it comes to writing papers or books, however, then it is much more difficult. I don't tend to write books. In papers I try to do as much as I can in writing an account and an introduction which gives the ideas. But you are committed to writing a proof in a paper, so you have to do that.
>
> Most books nowadays tend to be too formal most of the time, they give too much in the way of formal proofs, and not nearly enough in the way of motivation and ideas. …

> I think it is very unfortunate that most books tend to be written in this overly abstract way and don't try to communicate understanding.

Long (1986, 616):

> As far as the loss of certainty is concerned in itself (i.e., not as a historical-cultural phenomenon) it does not seem extraordinarily surprising or significant to me. I am far more puzzled by what "absolute certainty" might mean than by the fact that mathematics doesn't offer it. It seems to me that there is a similarity between this historical event (of 450 years duration) and the debating tactic which builds and then destroys a straw man.
>
> Mathematics is a human creation. That does not mean that it is arbitrary, but it does mean that it would be immodest to expect it to be "certainly true" in the common sense of the phrase. It is incoherent to try and imagine mathematics as a source or body of absolutely certain knowledge. . . .
>
> Proof is a form of mathematical discourse. It functions to unite mathematicians as practitioners of one mathematics. . . . a proof functions in mathematics only when it is accepted as a proof. This acceptance is a behavior of practicing mathematicians. . . .
>
> Fermat wrote that "the essence of proof is that it compels belief." To the extent that the compulsion operates via insight, (relatively) informal proofs will continue to play an important role in mathematics. Proofs that yield insight into the relevant concepts are more interesting and valuable to us as researchers and teachers than proofs that merely demonstrate the correctness of a result. We like a proof that brings out what seems to be essential. If the only available proof of a result is one that seems artifical or contrived it acts as an irritant. We keep looking and thinking. Instead of being able to move on, we are arrested. I mention these familiar facts only to emphasize that proof is not merely a system of links among various theorems, axioms, and definitions but also a system of discourse among people concerned with mathematics. As such it functions in a variety of ways.

From these first-hand informations we derive the following picture about the role of intuition and social agreement in elaborating and checking proofs in mathematics:

(1) The validity of a proof does not depend, at least not only, on its formal presentation within a formal axiomatic-deductive setting, but on the intuitive coherence of conceptual relationships and their agreement with the experiences of the researchers.

(2) The highly complex abstract theories of higher mathematics need a certain level of formal presentation for the sake of conceptual unambiguity and brevity. However, working with this formalism in a meaningful way presupposes an understanding of the communicative structures of the researchers working in the field and an intuitive understanding of the investigated objects. Any mathematical theory refers to a class of objects which can be represented in various ways and which via these representations become accessible for an operative study of their properties and relationships. Therefore mathematics is "quasi-empirical" (Lakatos 1963, 29 ff.; Jahnke 1978).

(3) Proofs serve primarily for understanding *why* the theorem in question is true. During the process of creating and sharing understanding among researchers proofs (and theorems!) are elaborated, re-formulated, generalized, improved, formalized etc. Along this process criter a of rigour may change. "Absolutely rigorous" proofs do not exist.

3 The Elementary-Mathematics-Research-Program of Mathematics Education

The changing views an proof in mathematics must be reflected in mathematics education in, as we believe, the following way:

(1) The teaching and learning of mathematics in the social context of school has to be based on a context of understanding and a frame of communication different from those in university mathematics. To appropriately transfer proof activities into the boundary conditions of school, we must abandon formal axiomatic-deductive presentations of relevant mathematical theories in favour of sound informal presentations. These are characterized by embeddings into meaningful contexts, by emphasis on motivation, by the use of heuristic strategies and pre-formal means of representation and by informal proofs. "Save the phenomena!" must be the maxim of mathematics education.

(2) Above all informal proofs can further understanding and therefore they have to be included into the *process* of learning and communication among students. Lakatos' "Proofs and Refutations" (Lakatos 1969) may serve as a model.

(3) The mathematical training of student teachers must contain informal courses in elementary mathematics in order to create a useful background for teaching. Comprehensive informal presentations of elementary mathematical theories are much more effective professional tools than background knowledge derived from formal presentations.

Within mathematics, formal presentations of elementary mathematics have a great tradition (cf. e.g. Lenz 1967; Griffith and Hilton 1976–1978) which we explicitly appreciate. But even if those presentations provide a lot of insight into mathematics teaching, from the point of view of this paper they are not sufficient. The literature of elementary mathematics, school mathematics, didactics of mathematics and the history of mathematics is full of informal approaches to certain problems, fields or even theories (cf. Sawyer 1964; Engel 1973/1976). To unify and to systematize these approaches, particularly by developing a "grammar" of iconic representations and concrete models, is in our view an extraordinarily important research problem of mathematics education which we would like to call the "elementary-mathematics-research-program" in mathematics education. The availability of comprehensive informal presentations of arithmetic, elementary algebra, elementary geometry, elementary stochastics and elementary analysis would lead to integrating mathematical, pedagogic, psychological and practical components of mathematics education and open a new level of didactical research and development and teacher training.

In order to indicate that this program points far beyond mathematics education we would like to close by quoting the mathematician Nowoshilow, member of the Soviet Academy of Science:

> The closed and sterile formal mathematics is not only a "luxury" which civilization can afford [as stated by Dieudonné] but also an inevitable consequence of civilization. From this point of view the fight against the spread of mathematical formalism among human beings around the world is an ecological task.
>
> (Epilogue in Blechman, Myschkis and Panovko 1984, 326).

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Chapter 6
Mathematics Education as a ‘Design Science’

Mathematics education (didactics of mathematics) cannot grow without close relationships to mathematics, psychology, pedagogy and other areas. However, there is the risk that by adopting standards, methods and research contexts from other well-established disciplines, the applied nature of mathematics education may be undermined. In order to preserve the specific status and the relative autonomy of mathematics education, the suggestion to conceive of mathematics education as a “design science” is made. In a paper presented to the twenty second Annual Meeting of German mathematics educators in 1988 Heinrich Bauersfeld presented some views on the perspectives and prospects of mathematics education. It was his intention to stimulate a critical reflection’among the members of the community’ on what they do and what they could and should do in the future (Bauersfeld 1988). The early seventies have witnessed a vivid programmatic discussion on the role and nature of mathematics education in the German speaking part of Europe (cf., the papers by Bigalke, Griesel, Wittmann, Freudenthal, Otte, Dress and Tietz in the special issue 74/3 of the *Zentralblatt für Didaktik der Mathematik* as well as Krygowska 1972). Since then the status of mathematics education has not been considered on a larger scale despite the contributions by Bigalke (1985) and Winter (1986). So the time is overdue for redefining the basic orientation for research; therefore, Bauersfeld’s talk could hardly have been more appropriate. In recent years the interest in a better understanding of the nature and role of mathematics education has also grown considerably at the international level as indicated, for example, by the ICMI-study on ‘What is research in mathematics education and what are its results?’ launched in 1992 (cf., Balacheff et al. 1992). The following considerations are intended both as a critical analysis of the present situation and an attempt to capture the specificity of mathematics education. Like Bauersfeld, the author presents them ‘in full subjectivity and in a concise way’ as a kind of ‘thinking aloud about our profession’. (The present paper concentrates on the didactics of mathematics although the line of

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E. C. Wittmann, *Connecting Mathematics and Mathematics Education*,
https://doi.org/10.1007/978-3-030-61570-3_6

argument pertains equally to the didactics of other subjects and also to education in general (cf., Clifford and Guthrie 1988, a detailed study on the identity crisis of the Schools of Education at the leading American universities).)

1 The 'Core' and the 'Related Areas' of Mathematics Education

> The sciences should influence the outside world only by an enlightened practice; basically they all are esoteric and can become exoteric only by improving some practice. Any other participation leads to nowhere.
>
> J.W. v. Goethe, *Maximen und Reflexionen*

Generally speaking, it is the task of mathematics education to investigate and to develop the teaching of mathematics at all levels including its premises, goals and societal environment. Like the didactics of other subjects mathematics education requires the crossing of boundaries between disciplines and depends on results and methods of considerably diverse fields, including mathematics, general didactics, pedagogy, sociology, psychology, history of science and others. Scientific knowledge about the teaching of mathematics, however, cannot be gained by simply combining results from these fields; rather it presupposes *a specific* didactic approach that *integrates* different aspects into a coherent and comprehensive picture of mathematics teaching and learning and then transposing it to practical use in a constructive way.

The specificity of this task necessitates, on the one hand, sound relationships to the disciplines related to mathematics education, and on the other hand, a balance between practical proximity and theoretical distance with respect to schools. Bauersfeld (1988, p. 15) refers here to the 'two cultures' of mathematics education. How we can integrate the variety of aspects, and at the same time, set weights and deal with the tensions that exist between theory and practice is not at all clear a priori. This is why it is so difficult to arrive at a generally shared conception of mathematics education.

In my view, the specific tasks of mathematics education can only be carried out if research and development have specific linkages with practice at their *core* and if the improvement of practice is merged with the progress of the field as a whole.

This *core* consists of a variety of components, including in particular:

- analysis of mathematical activity and of mathematical ways of thinking,
- development of local theories (for example, on mathematizing, problem solving, proof and practising skills),
- exploration of possible contents that focus on making them accessible to learners,
- critical examination and justification of contents in view of the general goals of mathematics teaching,
- research into the pre-requisites of learning and into the teaching/learning processes,

- development and evaluation of substantial teaching units, classes of teaching units and curricula,
- development of methods for planning, teaching, observing and analysing lessons, and
- inclusion of the history of mathematics education.

Work in the core necessitates the researcher’s interest and proximity to practical problems. A caveat is in order, however. The orientation of the core towards practice may easily lead to a narrow pragmatism that focuses on immediate applicability and may therefore become counterproductive. This hazard can only be avoided by connecting the core to a variety of related areas that bring about an exchange of ideas with related disciplines and that allow for investigating the different roots of the core in a systematic way (cf., Fig. 1). Of course, the core and the related areas overlap, and the ill-defined borders between them change over time. Thus, a strict separation is not possible.

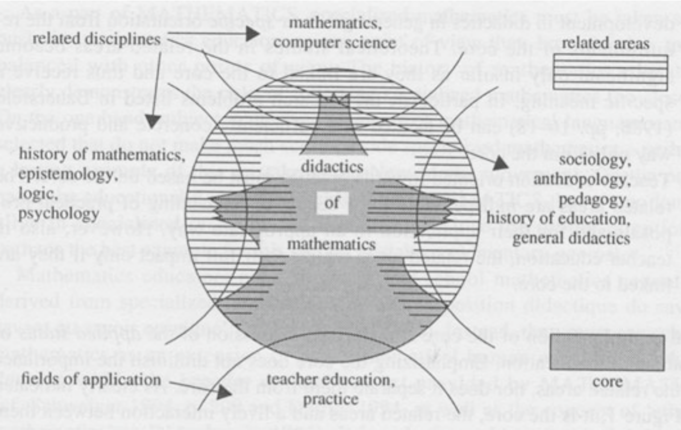

Fig. 1 The core and the areas related to mathematics education, their links to the related disciplines and the fields of application

Although the related areas are indispensable for the whole entity to function in an optimal way, the specificity of mathematics education rests on the core, and therefore the core must be the central component. Actually, progress in the core is the crucial element by which to measure the improvement of the whole field. This situation is comparable to music, engineering and medicine. For example, the composition and performance of music must take precedence over the history, critique and theory of music; in mechanical engineering the construction and development of machines is paramount to mechanics, thermodynamics and research of new materials; and in medicine the cure of patients is of central importance when compared to medical sociology, history of medicine or cellular research.

However, the division between the core and the related areas does not imply that the core is restricted to practical applications since the related areas have to develop the necessary theory. In fact, building theories or theoretical frameworks related to the design and empirical investigation of teaching is an essential component of work in the core (cf., Freudenthal 1987).

As in engineering, medicine and art, the different status of the core and the related areas is also clearly indicated in mathematics education by the following facts:

1. The core is aimed at an *interdisciplinary, integrative view of different aspects and at constructive developments* whereby the ingenuity of mathematics educators is of crucial importance. The related areas are derived much more from the corresponding disciplines. Therefore research and development in didactics in general get their *specific* orientation from the requirements of the core. Theoretical studies in the related areas become significant only insofar as they are linked to the core and thus receive a specific meaning. In particular, the research problems listed in Bauersfeld (1988, pp. 16–18) can be tackled in a sufficiently concrete and productive way only from the core.
2. Teacher education oriented towards practice must be based on the core. The related areas are indispensable for more deeply understanding practical proposals and their applications in an appropriate way. However, in teacher education too, the related areas realize their full impact only if they are linked to the core.

The central position of the core is mainly an expression of the *applied status* of mathematics education. Emphasizing the core does not diminish the importance of the related areas, nor does it separate them from the core. As clearly indicated in Fig. 1, it is the core, the related areas and a lively interaction between them that represent the full picture of mathematics education and that also necessitate the common responsibility of mathematics educators independent of their special fields of interest.

Work in the core must start from mathematical activity as an original and natural element of human cognition. Further, it must conceive of "mathematics" as a broad societal phenomenon whose diversity of uses and modes of expression is only in part reflected by specialized mathematics as typically found in university departments of mathematics. I suggest a use of capital letters to describe MATHEMATICS as mathematical work in the broadest sense; this includes mathematics developed and used in science, engineering, economics, computer science, statistics, industry, commerce, craft, art, daily life, and so forth according to the customs and requirements specific to these contexts. Specialized mathematics is certainly an essential element of MATHEMATICS, and the broader interpretation cannot prosper without the work done by these specialists. However, the converse is equally true: Specialized mathematics owes a great deal of its ideas and dynamics to broader scientific and societal sources. By no means can it claim a monopoly for "mathematics".

It should go without saying that MATHEMATICS, not specialized mathematics, forms the appropriate field of reference for mathematics education. In particular, the design of teaching units, coherent sets of teaching units and curricula has to be rooted in MATHEMATICS.

As a consequence, mathematics educators need a lively interaction with MATHEMATICS and they must devote an essential part of their professional life to stimulating, observing and analyzing genuine MATHEMATICAL activities of children, students and student teachers. Organizing and observing the fascinating encounter of human being with MATHEMATICS is the very heart of didactic expertise and forms a natural context for professional exchange with teachers.

As a part of MATHEMATICS, specialized mathematics must be taken seriously by mathematics educators as one point of view that, however, has to be balanced with other points of view. The history of mathematics education clearly demonstrates the risks of following specialized mathematics too closely: On the one hand, subject matter and elements of mathematical language can be selected that do not make much sense outside specialized mathematics—perhaps a lasting example of this mistake is the New Maths movement. On the other hand, the educationally important fields of MATHEMATICS that are no longer alive in specialized research and teaching may lose the proper attention—perhaps the best example for this second mistake is elementary geometry.

Mathematics educators must be aware that school mathematics cannot be derived from specialized mathematics by a “transposition didactique du savoir savant au savoir enseigné” (cf., Freudenthal 1986). Instead, they must see school mathematics as an extension of pre-mathematical human capabilities which develop within the broader societal context provided by MATHEMATICS (cf., Schweiger 1994: p. 299 and Dörfler 1994, as well as the concept of “ethno-mathematics” in D’Ambrosio 1986). It is only from this perspective that the unity of mathematics teaching from the primary through the upper secondary level can be established and that reasonable mathematical courses in teacher training can be developed which deserve to be called a scientific background of teaching.[1]

2 A Basic Problem in the Present Development of Mathematics Education: The Neglect of the Core

> The ‘hard sciences’ are successful, as they deal with ‘soft problems’. The ‘soft sciences’ are badly off, as they are confronted with ‘hard problems’.
>
> Heinz v. Foerster

An approach to the study of problems of learning and teaching in mathematics education requires a scientific framework that includes both research methods and standards. As a young discipline, mathematics education is under considerable pressure

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[1] I do not intend to give mathematical specialists advice not asked for. However, in my opinion it would also be beneficial for them to perceive themselves as partners in a larger mathematical system described by MATHEMATICS. Without some change of awareness on their part, all attempts to change the public image of “mathematics” are nothing but cosmetics and bound to fail.

from different directions. How to establish standards is as controversial as the status of mathematics education itself and can likewise be addressed in different ways.

One tempting approach is to adapt methods and standards from the hard sciences and the humanities. I dare say that all around the world quite a number of mathematics educators are taking this approach wherein the scientific background and their personal interests might be as influential as the wish to be recognized and supported by scientists in the related disciplines. However, approaches, methods and standards adopted from related disciplines are more easily applied to problems in the neighborhood of these disciplines than to problems in the core. Consequently, a great deal of didactic research adheres to mathematics, psychology, pedagogy, sociology, history of mathematics and so forth. Thus the holistic origin of didactic thinking, namely mathematical activity in social contexts, is dissolved into single strands, and the specific tasks of the core are neglected. In my view this is a big problem that presently inhibits major progress in mathematics education. The problem is by no means restricted to mathematics education, however. For example, Clifford and Guthrie (1988: p. 3) have identified it as a universal problem in education:

> Our thesis is that schools of education, particularly those located an the campuses of prestigious research universities have become ensnared improvidently in the academic and political cultures of their institutions and have neglected their own worlds. They have seldom succeeded in satisfying the scholarly norms of their campus letters and science colleagues, and they are simultaneously estranged from their professional peers. The more they have rowed toward the shores of scholarly research the more distant they have become from the public schools they are bound to serve.

The movement away from the core and towards the related areas may also be problematic because very often the adoption of frameworks and standards from related disciplines is linked to the dogmatic claim that these frameworks and standards were the only ones possible for didactics. From this position follows a blindness towards the central tasks of mathematics education and a systematic underestimation of the *constructive achievements* brought about in the core. Sometimes the core is even denied a scientific status. Mathematics educators who retreat into a "mathematical garden" (H. Meschkowski) tend of course to trivialize the educational aspects of mathematics education; similarly, those working in the areas related to psychology and pedagogy neglect the mathematical aspects. These tendencies are reinforced by voices from the related disciplines that argue against the scientific status of didactics more or less publicly. As a result we have an unreasonable set-back into reductionist positions analyzed as unfounded many years ago (cf., Bigalke 1985; Winter 1985). It is ironic that mathematics education set out in the late sixties to overcome exactly these polarized positions. What is urgently needed therefore is a methodological framework that does justice to the core of mathematics education.

3 Mathematics Education as a Systemic-Evolutionary ‘Design Science’

> It is the yardstick that creates the phenomena... A religious phenomenon can only be revealed as such if it is captured in its own modality, i.e., if it is considered by means of a religious yardstick. To locate such a phenomenon by means of physiology, psychology, sociology, economics, linguistics, art, etc. means to deny it. It means to miss exactly its uniqueness and its irreducibility.
>
> Mircea Eliade, *The Religions and the Sacred*

Establishing[2] scientific standards in mathematics education by adopting standards from related disciplines is, as mentioned, unwise because problems and tasks of mathematics education tend to be tackled only insofar and to the extent that they are accessible to the methods of the related disciplines. As a consequence, the core is not sufficiently recognized as a scientific field in its own right.

Fortunately there is a silver lining in this dilemma if one abandons the fixation on the traditional structures of the scientific disciplines and instead looks at the specific character of the core, namely the *constructive* development of and research into mathematics teaching. Here mathematics education is assigned to the larger class of “design sciences” (cf., Wittmann 1974) whose scientific status was clearly delineated from the scientific status of natural sciences by the Nobel Prize Winner Herb Simon. The following quotation from Simon (1970, pp. 55–58) explains also the resistance offered to the design sciences in academia. In this way the present situation of mathematics education is embedded into a wider context and becomes accessible to a rational evaluation.

> Historically and traditionally, it has been the task of the science disciplines to teach about natural things: how they are and how they work. It has been the task of engineering schools to teach about artificial things: how to make artifacts that have desired properties and how to design …
>
> Design, so construed, is the core of all professional training; it is the principal mark that distinguishes the professions from the sciences. Schools of engineering, as well as schools of architecture, business, education, law and medicine, are all centrally concerned with the process of design.
>
> In view of the key role of design in professional activity, it is ironic that in this century the natural sciences have almost driven the sciences of the artificial from professional school curricula. Engineering schools have become schools of biological science; business schools have become schools of finite mathematics …

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[2] The term “design” and related terms used subsequently in this paper might cause irritation, for in traditional understanding these terms are linked to mechanistic procedures of making tools and controlling systems (cf., Jackson 1968: 163 ff.). In part 3 of this paper we will show, however, that in striking contrast to the “mechanistic” paradigm of design and management there is a new “systemic-evolutionary” paradigm based on the appreciation of the complexity and self-organisation of living systems. It is in the context of this new paradigm that the term “design” and similar ones are used in the present paper.

> The movement toward natural science and away from the sciences of the artificial has proceeded further and faster in engineering, business and medicine than in the other professional fields I have mentioned, though it has by no means been absent from schools of law, journalism and library science …
>
> Such a universal phenomenon must have a basic cause. It does have a very obvious one. As professional schools …are more and more absorbed into the general culture of the university, they hanker after academic respectability. In terms of the prevailing norms, academic respectability calls for subject matter that is intellectually tough, analytic, formalizable and teachable. In the past, much, if not most, of what we knew about design and about the artificial sciences was intellectually soft, intuitive, informal and cookbooky. Why would anyone in a university stoop to teach or learn about designing machines or planning market strategies when he could concern himself with solid-state physics? The answer has been clear: he usually wouldn't …
>
> The older kind of professional school did not know how to educate for professional design at an intellectual level appropriate to a university; the new kind of school has nearly abdicated responsibility for training in the core professional skills …
>
> The professional schools will reassume their professional responsibilities just to the degree that they can discover a science of design, a body of intellectually tough, analytic, partly formalizable, partly empirical, teachable doctrine about the design process.
>
> It is the thesis of this chapter that such a science of design not only is possible but is actually emerging at the present time.[3]

In the writer's opinion the framework of a design science opens up to mathematics education a promising perspective for fulfilling its tasks and also for developing an unbroken self-concept of mathematics educators. This framework supports the position described in part 2, for the core of mathematics education concentrates on constructing "*artificial objects*", namely teaching units, sets of coherent teaching units and curricula as well as the investigation of their possible effects in different educational "ecologies". *Indeed the quality of these constructions depends on the theory-based constructive fantasy, the "ingenium", of the designers, and on systematic evaluation, both typical for design sciences.* How well this conception of mathematics education as a design science reflects the professional tasks of teachers

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[3] The underestimation of the "skills of designing and making" is deeply rooted in our culture. Cf. A. Smith, A coherent set of decisions, the Stanley Lecture, Manchester Polytechnic 1980: p. 22:

> Throughout the whole of our society we show little respect for the skills of designing and making. Indeed in many of our schools these very skills are looked down upon and are referred to as the noddy subjects, fit only for the less able in our community.
>
> I remember, during my years as chairman of the Schools Council, visiting a school where, after I had been shown the fairly conventional range of school work, I was taken into the workshops and there on the bench was a most beautiful and competent piece of metal work. It was a joy to look at, but it was described to me as a piece of work "by one of our less able pupils". It was an extraordinary description, which spoke volumes about our distorted scale of values. There was a piece of work which expressed ability, as fine in its way as the best essay written by the highest flyer in English, but never seen by academic people as such. To write things with pen on paper is an up-marked, respectable activity; to conceive pattern in your mind and to make them with your hands is a down-marked activity, less worthy of respect.

is shown, for example, by Clark and Yinger (1987, pp. 97–99) who have identified teaching as a "design profession".

The clear structural delineation of mathematics education as a *design* science from the related sciences underlines its specific character and its relative independence. Mathematics education is not an appendix to mathematics, nor to psychology, nor to pedagogy for the same reason that any other design science is not an appendix to any of its related disciplines. Attempts to organize mathematics education by using related disciplines as models miss the point *because they overlook the overriding importance of creative design for conceptual and practical innovations.*

As far as research frameworks and standards are concerned, mathematics educators working in the core should primarily start from the achievements in the core already available. There is no doubt that during the past 25 years significant progress, that includes the creation of theoretical frameworks, has been made within the core and that standards have been set which are well-suited as an orientation for the future. "Developmental research" as suggested by Freudenthal and elaborated by Dutch mathematics educators is a typical example (cf., Freudenthal 1991, pp. 160–161; and Gravemeijer 1994). Of course, it is reasonable also to adopt methods and standards from the related disciplines to the extent that they are appropriate to the problems of the core.

It is no surprise that there objections to the view of mathematics education as a "design science" emerge, for the simple reason that the design sciences have traditionally followed—and are still widely following—a mechanistic paradigm whose harmful side effects are becoming more and more visible. This approach would certainly be detrimental to education. However, we are presently witness to the rise of a new paradigm for the design sciences that is based on the "systemic-evolutionary" development of living systems and takes the complexity and self-organization of these systems into account (cf., Malik 1986). Even if researchers in the design sciences in general hesitate to adopt this new paradigm, there is no reason why mathematics educators should not follow it, even more so since this paradigm corresponds to recent developments in the field. The systemic-evolutionary view on the teacher-student and the theorist-practitioner relationships differs greatly from the traditional view. Knowledge is no longer seen as the result of a transmission from the teacher to a passive student, but is conceived of as the productive achievement of the student who learns in social interaction with other students and the teacher. Therefore the materials developed by mathematics educators must be construed so as to acknowledge and allow for this interactive approach. In particular, they must provide teachers and students the freedom to make choices of their own. In order to facilitate and stimulate a flexible use of the materials designed in this way, teachers have to be trained and regarded as partners in research and development and not as mere recipients of results (cf. Schupp 1979; Schwab 1983; Fischer and Malle 1983, and the papers by Brown/Cooney, Seeger/Steinbring, Voigt, and others in *Zentralblatt für Didaktik der Mathematik* (4/91 and 5/91)). As a consequence, teacher training receives a new quality. An important orientation for innovations along these lines is the approach developed by Schön (1987) for the training of engineers that is based upon the idea of the "reflective practitioner".

As a systemic-evolutionary design science mathematics education can follow different paths. It is certainly not reasonable to develop it into a "monoparadigmatic" form as postulated, for example, for the natural sciences. In a design science the simultaneous appearance of different approaches is a sign of progress and not of retardation as stated by Thommen (1983, p. 227) for management theory:

> Because of a continuously changing economic world it is possible to (re-)construct an economic context within different formal frameworks, or models. These need not be mutually exclusive, on the contrary, they can even be complementary, for no model can take all problems and aspects into account as well as consider and weigh them equally. The more models exist, the more problems and aspects are studied, the greater is the chance for mutual correction. Therefore we consider the variety of models in management theory as an indicator for an advanced development of this field moving on in an evolutionary, not a revolutionary process in which new models emerge and old ones disappear.

4 The Design of Teaching Units and Empirical Research

> That, in concrete operation, education is an art, either a mechanical art or a fine art, is unquestionable. If there were an opposition between science and art, I should be compelled to side with those who assert that education is an art. But there is no opposition, although there is a distinction.
>
> John Dewey, *On the sources of a science of education*

For developing mathematics education as a design science it is crucial to find ways how design on the one hand and empirical research on the other can be related to one another. In the following the writer proposes a specific approach to empirical research, namely empirical research centered around teaching units.

It cannot be denied that teaching units, and on a wider scale curricula, have found attention in mathematics education in the past. In fact, curriculum development held a prominent place in the late sixties and early seventies. Nevertheless, the writer contends that the design of teaching units has never been a focus of research. At best teaching units have been used as more or less incidental examples in investigating and presenting theoretical ideas. Many of the best units were published in teachers' journals, not in research journals, and were hardly noticed by the research community. For this phenomenon the following explanation is offered: In contrast to "research", the design of teaching has been considered as a mediocre task normally done by teachers and textbook authors. To rephrase Herb Simon: Why should anyone anxious for academic respectability stoop to designing teaching and put him- or herself on one level with teachers? The answer has been clear: He or she usually wouldn't.

In order to overcome this fundamentally incorrect view we have to recognize that in all fields of design there is—by the very nature of design—a wide spectrum of competence and experience ranging from the amateur, to the novice, the less or more skilled worker, the experienced master, up to the creative inventor. Typically, the bulk of design on a larger scale is done in special centers for research and development.

As a design science mathematics education can be no exception from this rule. That teachers take part in design can be no excuse for mathematics educators to refrain from this task. On the contrary: The design of substantial teaching units, and particularly of substantial curricula, is a most difficult task that must be carried out by the experts in the field. By no means can it be left to teachers, although teachers can certainly make important contributions within the framework of design provided by experts, particularly when they are members of or in close connection with a research team. Also, the adaptation of teaching units to the conditions of a special classroom requires design on a minor scale. Nevertheless, a teacher can be compared more to a conductor than to a composer or perhaps better to a director ("metteur en scène") than to a writer of a play. For this reason there should exist strong reservations about "teachers' centers" wherein teachers meet to make their own curriculum.

We should be anxious to delineate teaching units of the highest quality from the mass of units developed at various levels for various purposes. These "substantial" teaching units can be characterized by the following properties:

1. They represent central objectives, contents and principles of mathematics teaching.
2. They provide rich sources for mathematical activities.
3. They are flexible and can easily be adapted to the conditions of a special classroom.
4. They involve mathematical, psychological and pedagogical aspects of teaching and learning in a holistic way, and therefore they offer a wide potential for empirical research.

Typically, a substantial teaching unit always carries a name. As examples I mention "Arithmogons" by Alistair McIntosh and Douglas Quadling, "Mirror cards" by Marion Walter, "Giant Egbert" and other units developed in the Dutch Wiskobas project, and Gerd Walther's unit "Number of hours in a year" (Walther 1984, 72–78). Other examples and a systematic discussion of the role of substantial teaching units in mathematics education are given by Wittmann (1984).

For the sake of clarity, one example of a substantial teaching unit is sketched below. In our primary project Mathe 2000 the following setting of arithmogons is used in grade 1:

A triangle is divided in three fields by connecting its midpoint to the midpoints of its sides. We put counters or write numbers in the fields. The simple rule is as follows: Add the numbers in two adjacent fields and write the sum in the box of the corresponding side (cf. Fig. 2).

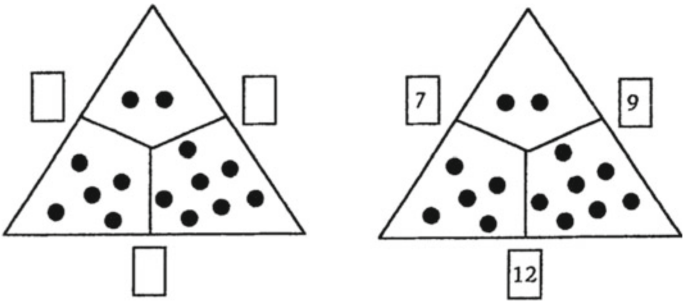

Fig. 2 Modified representation of arithmogons

Various problems arise: When starting from the numbers inside, the numbers outside can be obtained by addition. When one or two numbers inside and respectively two or one number outside are given, the missing numbers can be calculated by addition and subtraction. When the three numbers outside are given, we have a problem that does not allow for direct calculation but requires some thinking. It turns out that there is always exactly one solution. However, it may be necessary to use fractions or negative numbers.

The mathematics behind arithmogons is quite advanced: The three numbers inside form a vector as well as the three numbers outside. The rule of adding numbers in adjacent fields defines a linear mapping from the three-dimensional vectorspace over the reals into itself. The corresponding matrix is non-singular. One can generalize the structure to n-gons as shown in McIntosh and Quadling (1975).

The teaching unit based upon arithmogons consists of a sequence of tasks and problems that arise naturally from the mathematical context. The script for the teacher may be structured as follows:

1. Introduce the rule by means of examples and make sure that the rule is clearly understood.
2. Present some examples in which the numbers inside are given.
3. Present some examples in which some numbers inside and some numbers outside are given.
4. Present a problem in which the numbers outside are given.
5. Present other problems of this kind.

As can be seen, a substantial teaching unit is essentially open. Only the key problems are fixed. During each episode the teacher has to follow the students' ideas in trying to solve the problems. This role of the teacher is completely different from traditional views of teaching. Teaching a substantial unit is basically analogous to conducting a clinical interview during which only the key questions are defined and the interviewer's task is to follow the child's thinking.

The structural similarity between substantial teaching units on the one hand, and clinical interviews on the other, suggests an adaptation of Piaget's method for studying children's cognitive development to empirical research on teaching units (cf., Fig. 3). As a result we arrive at "clinical teaching experiments" in which teaching units can be used not only as research tools, but also as objects of study.

	Tools	**Method**
Piagetian Psychology	Structured sets of tasks	Clinical interviews
Mathematics Education	Teaching units	Clinical teaching experiments

Fig. 3 Comparison of clinical interviews with teaching experiments

The data collected in these experiments have multiple uses: They tell us something about the teaching/learning processes, individual and social outcomes of learning, children's productive thinking, and children's difficulties. They also help us to evaluate the unit and to revise it in order to make teaching and learning more efficient.

The Piagetian experiments were repeated many times by other researchers. Many became a focus of extended psychological research. Some even established special lines of study; for example, the "conservation" experiments. It is no exaggeration to say that Piaget's experiments and the patterns he observed in children's thinking survived much longer than his theories, in many cases until the present. In the same way, clinical teaching experiments can be repeated and thereby varied. By comparing the data we can identify basic patterns of teaching and learning and derive well-founded specific knowledge on teaching certain units. Much can be learned here from Japanese research in mathematics education (cf., Becker and Miwa 1989).

In conducting such studies, existing methods of qualitative research can effectively be used, particularly those developed by French mathematics educators in connection with "didactical situations" and with "didactic engineering" (cf. Brousseau 1986; Artigue and Perrin-Glorian 1991; Arsac et al. 1992). Concerning the reproducibility of results it is very instructive to look at the social sciences. Friedrich von Hayek, another Nobel prize winner in economics, has convincingly pointed out that empirical research on highly complex social phenomena yields reproducible results if directed towards revealing general patterns beyond special data (von Hayek 1956). To admit that the results of teaching and learning depend on the students and on the teacher does not preclude the existence of patterns related to the mathematical content of a specific teaching unit (cf., also, Kilpatrick 1993, pp. 27–29 and Sierpinska 1993, pp. 69–71). Of course, we must not expect all these patterns to arise on any occasion nor under all circumstances. It is quite natural that patterns will occur, varying with the educational ecologies. One should be reminded here of the well-known fact that

Piagetian interviews also reveal recurring content-specific patterns which, however, do not occur with every individual child.

Research centered around teaching units is useful for several reasons. First, it is related to the subject matter of teaching (cf., the postulate of "relatedness" in Kilpatrick 1993, p. 30). Second, knowledge obtained from clinical teaching experiments is "local". Here we need to be more careful in generalizing over contents than we have been in the past. In the future we can certainly expect to derive theories covering a wide range of teaching and learning. But these theories cannot emerge before a variety of individual teaching units has been investigated in detail. For studying the mathematical theory of groups the English mathematician Graham Higman stated in the fifties "that progress in group theory depends primarily on an intimate knowledge of a large number of special groups". The striking results achieved in the eighties in the classification of finite simple groups showed that he was right. In a similar way, the detailed empirical study of a large number of substantial teaching units could prove equally helpful for mathematics education.

Third, theory related to teaching experiments is meaningful and applicable. We should, however, be aware that, due to the inherent complexity of teaching and learning, the data and theories that research might provide may never provide complete information for teaching a certain unit. Only the teacher is in a position to determine the special conditions in his or her classroom. Therefore there should be no sharp separation between the researcher and the teacher as stated earlier. As a consequence, teachers have to be equipped with some basic competence in doing research on a small scale. The writer's experience in teacher training indicates that introducing student teachers into the method of clinical interviews is an excellent way towards that end (Wittmann 1985).

In the writer's opinion, the most important results of research in mathematics education are sets of carefully designed and empirically studied teaching units that are based on fundamental theoretical principles. It follows that these units should form a major part of the professional training of teachers. Teachers who leave the university should have in their baggage a set of substantial teaching units that represent the standards of teaching. From the experiences with our primary project Mathe 2000 it is clear that such units are the most efficient carriers of innovation and are well-suited to bridge the gap between theory and practice.

5 And the Future of Mathematics Education?

> The frogs tend to forget that once they were tadpoles, too.
> *Korean Proverb*

Generally speaking, it may be taken for granted that dealing in an intelligent way with complex systems on a scientific basis will become inevitable in all parts of human life. Very often the methods offered by the specialized disciplines are not sufficient. Riedel (1988) recently pleaded for a more context-related, more practical and less-formal "second philosophy", in contrast to the traditional "first philosophy" that aims at complete descriptions and deductions and that is bound to fail when applied to complex systems, because of its "ideology of self-restriction" (Fischer 1980). This seems to be a signal for a critical reflection in all sciences from which mathematics education as a systemic-evolutionary design science can take profit in the long range, since society will have to accept the fact that the development of human resources is at least as important for economic prosperity as is the development of new technologies and new marketing strategies.

In the short run the status of didactics in the universities will remain arduous. The resistance from the specialized sections within the related disciplines to establishing didactics in teacher training programs at *all* levels and to funding research in didactics is likely to continue. The history of the universities shows many instances in which scholars of established disciplines displayed their ignorance and acted in an unfair way towards newly evolving disciplines. The resistance of the old universities towards the technical schools at the end of the 19th century, the resistance of pure mathematicians towards applied ones at the beginning of this century and the vote of the German Philosophical Society against the establishment of chairs of pedagogy at the universities in the fifties are only a few examples. Obviously it is difficult, if not impossible, for specialists to understand and to appreciate new developments on the very borderline of their discipline.

In order to strengthen their position at the universities and to acquire funds from research foundations, mathematics educators need support from society. In this respect the relationships of mathematics education to the schools play a fundamental role. The use and the indispensability of didactic research for improving practice have to be convincingly demonstrated to teachers, supervisors, administrators, parents and the public. This can only be achieved from the core, that is, by concentrating on central tasks and by organizing design, empirical research and teacher education accordingly.

At the same line, there is potential in establishing a network of "Public—School—School Administration—Teachers' Unions—Teacher Training—Design, Research, Development" people in which the core of mathematics education will naturally find its proper place. In other words, organizing a systemic effort involving all the constituent groups.

This is consistent with the advice given by Clifford and Guthrie to schools of education in general (cf., Clifford and Guthrie 1988: pp. 349–350):

> The major mission of schools of education should be the enhancement of education through the preparation of educators, the study of the educative process, and the study of schooling as a social institution. As John Best has observed, the challenge before schools of education is quite different from that confronting the specialist in politics in a department of political science; concerned with building the discipline, he or she is under no obligation to train country clerks, city managers, and state legislators, and to improve their performance by conducting research directed toward that end. In order to accomplish *their* charter, however, schools of education must take the profession of education, not academia, as their main point of reference. It is not sufficient to say that the greatest strength of schools of education is that they are the only places available to look at fundamental issues from a variety of disciplinary perspectives. They have been doing so for more than half a century without appreciable effect on professional practice. It is time for many institutions to shift their gears.

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Chapter 7
Designing Teaching: The Pythagorean Theorem

1 Introduction

Here we consider a fundamental activity of teachers that has to be based on an integrated view of mathematics and pedagogy in order to be successful: namely, the design of teaching. As expressed in the title, a well-known topic of geometry, the Pythagorean theorem, is used for illustrating this integrative approach to student teachers. In other words, the emphasis of the paper is *less* on the Pythagorean theorem per se but *more* on general principles of a teacher's "design kit" that can be applied to other topics as well.

The design of teaching lies at the very heart of a teacher's professional activities. That is why some authors conceive of teaching as a design profession and correspondingly of mathematics education as a *design science* (Clark and Yinger 1987; Wittmann 1985, 1995). For this reason the paper can also be understood as an example of how to organize research and development in mathematics education along the lines of design.

Accordingly there are four sections that follow. The first section is to make the reader think about the Pythagorean theorem within the context of school by remembering personal experiences from school and university; by solving textbook problems; by looking at the treatment of the Pythagorean theorem in textbooks; and by interviewing students on what they have retained from teaching.

The second section introduces the reader to the framework of mathematical concepts behind and around this theorem and its proofs; problem contexts from which the theorem naturally arises; and research on students' psychological development in understanding and using these concepts.

The third section will demonstrate how the mathematical, heuristic, and psychological strands from the second section have to be related and tuned to one another, that is to be *integrated*, in the design of teaching units. The section contains teaching plans of introductory teaching units for the Pythagorean theorem.

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The final section explains some key concepts that can be generalized from the three strands of Sect. 2: the notion of "informal" proof; the heuristic strategy "specializing"; and the so-called operative principle.

All three principles will also be illustrated by subject matter different from the Pythagorean theorem in order to stimulate the transfer of these key concepts to other topics.

2 Thinking About the Pythagorean Theorem within the School Context

> But neither thirty years, nor thirty centuries, affect the clearness, or the charm, of geometric truths. Such a theorem as "the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the sides" is as dazzlingly beautiful now as it was in the days when Pythagoras first discovered it, and celebrated its advent, it is said, by sacrificing one hundred oxen – a method of doing honour to science that has always seemed to me slightly exaggerated and uncalled for. One can imagine oneself, even in these degenerate days, marking the epoch of some brilliant scientific discovery by inviting a convivial friend or two, to join one in a beefsteak and a bottle of wine. But one hundred oxen! It would produce a quite inconvenient supply of beef.
>
> C.L. Dodgson

In any right triangle the area of the square described on its longest side (the hypotenuse) is equal to the sum of the areas of the squares described on the other two sides (the legs).

This theorem is named after the Greek philosopher Pythagoras who lived around 500 B.C. and was the spiritual leader of a kind of philosophic-religious sect (the Pythagorean brotherhood, see van der Waerden 1978). Historians are certain that the fact stated in the theorem was already known to the ancient Babylonians, Egyptians and Chinese. So Pythagoras did not discover it, but might have been one of the first to give a proof.

The Pythagorean theorem enables one to compute the length of the third side of a right triangle if the lengths of the other two sides are given. In elementary geometry and its applications this situation arises very frequently when informations about lengths of segments are near at hand and right triangles can easily be identified or introduced.

Because of its richness in mathematical relationships and applications the Pythagorean theorem and its generalizations form a cornerstone of geometry. Mathematicians do not hesitate to rank the theorem among the top 20 theorems of all times. Without any doubt the Pythagorean theorem is *the* outstanding theorem of school mathematics. Generations of students have learned it, willingly or unwillingly, and many of them have kept the "Pythagorean" in their mind throughout their lives as the incarnation of a mathematical theorem.

Before interacting with the views expressed in this paper it is absolutely necessary for you first to mobilize your knowledge about the Pythagorean theorem and to get some fresh first hand experiences about the Pythagorean theorem, its teaching and, most important, about the learners. The following six activities are intended as catalysts for "jumping in."

Hints to solutions can be found in the appendix, but first try yourself.

Exploration 1

Write down your own "memories" of the Pythagorean theorem both from school and university. Do you remember how the theorem was introduced, proved, applied? Did you encounter the theorem later on? Discuss your notes with your fellow students.

Exploration 2

Fig. 1 shows a cartoon from the nineteenth century. Discuss it in terms of the Pythagorean theorem: What special case is represented and how can it be proved from the two shapes?

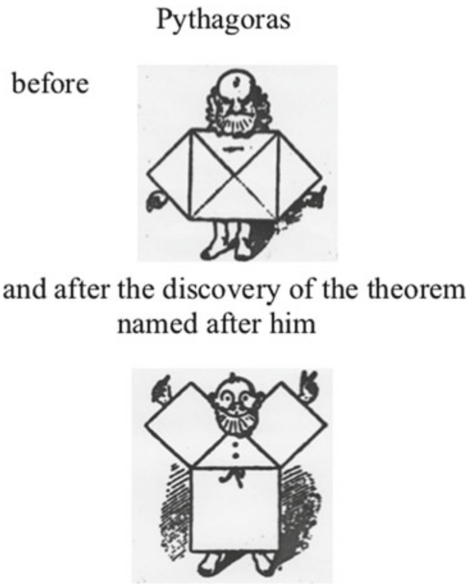


Fig. 1

Exploration 3

The following three problems may serve as a test for your feeling about the appropriate use of the Pythagorean Theorem.
Solve the problems, and record whether or not you used the Pythagorean theorem.

1. How long is the spatial diagonal s in a rectangular solid with edges a, b, c (see Fig. 2)?

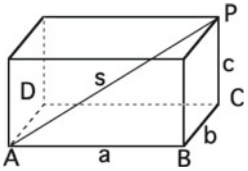


Fig. 2

2. The vertices of a square and the midpoints of its sides are connected as shown in Fig. 3. What part of the area is formed by the shaded figure?

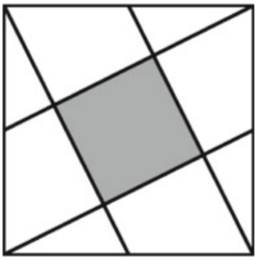


Fig. 3

3. A car is jammed in a parking lot. Under which conditions is it possible for the car to move out of the lot? Represent cars by pieces of cardboard, do some tests and devise a geometric model (see Fig. 4).

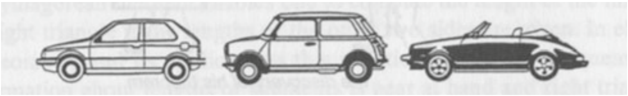


Fig. 4

Exploration 4

Because of its prominent role in school mathematics the Pythagorean theorem provides a rich source for collecting data on what "remains" in students after they have been taught the Pythagorean theorem in school.

The following interview form (Fig. 5) may give you an idea of how to probe students' thinking about the Pythagorean theorem. The interview starts with questions that scratch only the surface of the Pythagorean theorem and from there goes on to questions that test understanding.

*Age:*__________ *Intended profession:*____________________

1. Do you remember the Pythagorean theorem and can you write it down?
2. Do you have an idea of what the Pythagorean theorem is good for?
3. Can you give an example for its use in some profession?
4. Can you relate Fig. 5 to the Pythagorean theorem?
5. Do you know a proof of the Pythagorean theorem?

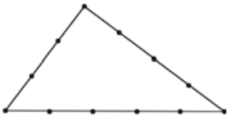


Fig. 5

1. Use the above interview form (or make your own form) and interview some students from grades 9 to 12. You may also ask some students to give written responses.
2. Analyze your data. Are there recurring patterns in students' responses?

Exploration 5

Select a sample of textbooks for grades 7 to 10 and investigate if and how the Pythagorean theorem is introduced, proved, and applied. Which approach do you find most convincing? Discuss your choice with your fellow students.

Exploration 6

If you had to design a teaching unit for introducing the Pythagorean theorem on the basis of your present knowledge about the Pythagorean theorem, what basic idea would you choose and why?

3 Understanding the Structure of the Pythagorean Theorem

The design of teaching units requires a thorough understanding of the subject matter and of the psychological premises for learning it as teaching is a continuous process of mediating between the mathematical structure of the subject matter and the cognitive structures of the learners.

The best way to understand the *mathematical* structure of the Pythagorean theorem consists of examining proofs of and heuristic approaches to it. In order to get information about the *psychological* structures on which the teaching of this theorem can be based we have to look into developmental research on students' thinking about basic notions relevant in this context.

Although the mathematical, the heuristic, and the psychological strands of the Pythagorean theorem will be investigated separately in this section and their proper integration is to be attacked in the next section within the design of teaching units, relationships between them will become apparent quite naturally without taking special effort.

3.1 Different Proofs of the Pythagorean Theorem

> The main goal of all science is first to observe and then to explain phenomena. In mathematics the explanation is the proof.
>
> D. Gale

The richness of the Pythagorean theorem in conceptual relationships is clearly demonstrated by a multitude of different proofs. Lietzmann (1912) lists about 20 proofs, Loomis (1968) in his classic "The Pythagorean Proposition" even 370, most of which, however, are obtained from a few basic proofs by slight variations. It is interesting to realize that the Pythagorean theorem is rooted in all cultures. A particularly nice ethnomathematical approach based on a special decorative motif was developed by Gerdes (1988).

The following four proofs and their variations are interesting for both historic and educational reasons, and they cover also the essential approaches to the Pythagorean theorem found in textbooks. These four proofs are presented here as they are typically met in the mathematical literature. Taken as they stand they certainly cannot serve as a model for lively teaching, and the reader might wonder why they have been included here. However, the proofs display the conceptual relationships behind and around the Pythagorean theorem in the most effective way, and so analyzing and comparing them is indispensable for integrating content and pedagogy. Moreover, it will be instructive for the reader to compare the "lecture style" of this section with the process-oriented style of the next section and to see by what means life can be brought into seemingly "dead" content.

Proof 1 (*Euclid's proof (Euclid, Book I, §47)*) Euclid's famous *Elements of Mathematics* (1926) represents the first systematic mathematical treatise ever written. The thirteen books develop elementary geometry and arithmetic through a deductively organized sequence of theorems and definitions starting from basic concepts and axioms. The *Elements* has been the most influential mathematical textbook of all times and up to the twentieth century has also determined the teaching of geometry at school.

At the end of Book I we find the Pythagorean theorem (Proposition 47, see Fig. 6):

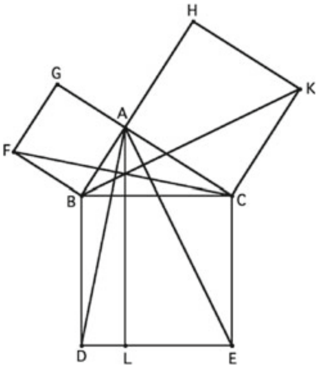


Fig. 6

> In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

> Let ABC be a right-angled triangle having the angle BAC right; I say that the square on BC is equal to the squares on BA, AC.
> For let there be described on BC the square $BDEC$, and on BA, AC the squares GB, HC; through A let AL be drawn parallel to either BD or CE, and let AD, FC be joined. Then, since each of the angles BAC, BAG is right, it follows that with a straight line BA, and at the point A on it, the two straight lines AC, AG not lying on the same side make the adjacent angles equal to two right angles; therefore CA is in a straight line with AG.
> For the same reason BA is also in a straight line with AH. And, since the angle DBC is equal to the angle FBA: for each is right: let the angle ABC be added to each; therefore the whole angle DBA is equal to the whole angle FBC.
> And, since DB is equal to BC, and FB to BA, the two sides AB, BD are equal to the two sides FB, BC respectively; and the angle ABD is equal to the angle FBC; therefore the base AD is equal to the base FC, and the triangle ABD is equal to the triangle FBC.
> Now the parallelogram BL is double of the triangle ABD, for they have the same base BD and are in the same parallels BD, AL.
> And the square GB is double of the triangle FBC, for they again have the same base FB and are in the same parallels FB, GC. (But the doubles of equals are equal to one another.)
> Therefore the parallelogram BL is also equal to the square GB.
> Similarly, if AE, BK be joined, the parallelogram CL can also be proved equal to the square HC; therefore the whole square $BDEC$ is equal to the two squares GB, HC.

> And the square $BDEC$ is described on BC, and the squares GB, HC on BA, AC. Therefore the square on the side BC is equal to the squares on the sides BA, AC.
> Therefore etc.
>
> Q.E.D.

Proof 1* (*Dynamic Version of Euclid's Proof*) In order to seek a more palatable way of understanding a proof whose diagram is as complicated as Euclid's proof, it is necessary first of all to understand the essence of the proof. What is Euclid trying to do? He has two squares, $BAGF$ and $ACKH$, on the sides of the right triangle ABC. He wants to show that the sum of the areas of these squares is equal to square $BCED$, the one on the hypotenuse. How does he do that? We can reduce the number of lines considerably for the purpose of demonstrating what he is trying to show (Fig. 7).

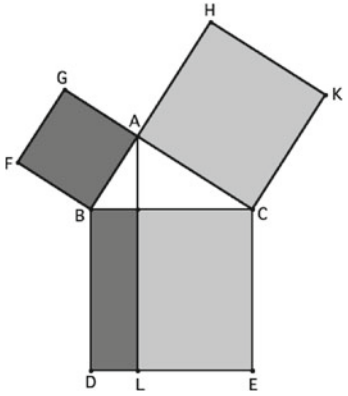


Fig. 7

He is trying to demonstrate that the area of square $BDEC$ can actually be decomposed into two pieces, one equal in area to square $BAGF$ and the other equal in area to $ACKH$. He demonstrates that the two darker regions are equal in area and the two lighter regions are equal in area.

Once you have convinced yourself that the above description is an accurate rendition of Euclid's proof, then you are in a position to create a proof that has more visual appeal. One such proof involves transforming each of the small squares on the sides of the original triangle into something more dynamic than the triangles as intermediaries. We can actually imagine the original small squares being transformed progressively into several parallelograms before actually forming the shaded rectangles that compose square $BCDE$.

Once we have the essence of the proof, we are still left with the pedagogically interesting task of transforming something quite technical and nonintuitive into something that is dynamic and intuitive. Euclid's proof shows that the two lighter regions are equal in area by introducing an intermediary figure: $\Delta\ FBC$. He shows that the two darker regions are equal in area by introducing another intermediary figure: $\Delta\ BCK$. Each member of the pair of similarly shaded regions is equal to twice the area of that corresponding triangle.

Below is a description of the stages of successive transformation (see Fig. 8).

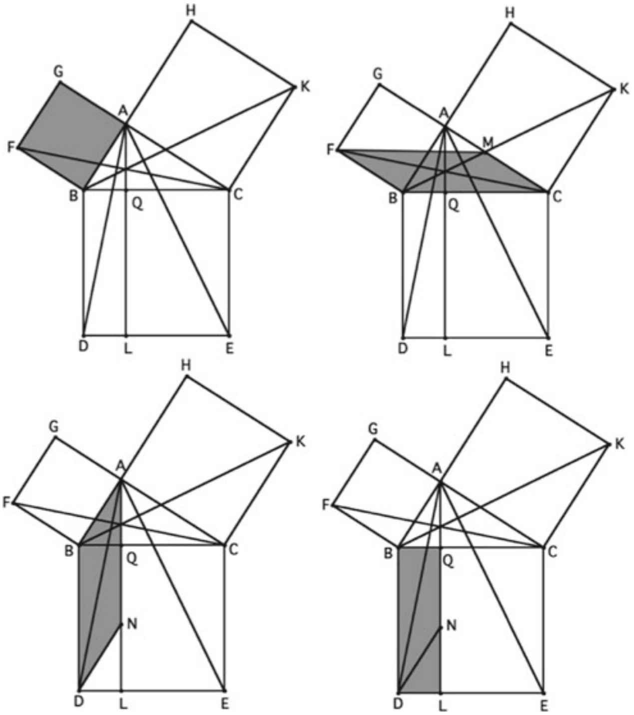


Fig. 8

1. Square $BAGF$ is sheared into parallelogram $BCMF$.
2. Parallelogram $BCMF$ is rotated into parallelogram $BDNA$.
3. Parallelogram $BDNA$ is sheared into rectangle $BDLQ$.

All three transformations preserve area. Therefore square BAGF and rectangle BDLQ have equal areas. In an analogous way square $ACKH$ is transformed into rectangle $QLEC$. As a consequence the area of $CBDE$ is equal to the sum of the areas of squares $ACKH$ and $BAGF$.

It is tempting to reduce the whole argument to a film simply "showing" the equality of areas. However, this would give a distorted view of proof. A visual demonstration can certainly support, but not replace, a proof. The proof hinges upon a conceptual framework that explains *why* there transformations can be applied and *why* they lead to the properties in question.

Exploration 7

Compare proofs 1 and 1*. Which parts of proof 1 correspond to which parts of 1*? Are there details in proof 1 that are missing in proof 1*? What are the advantages and the disadvantages of the formal language of "signs" in proof 1 and the informal language of "pictures" in proof 1*?

While Proofs 1 and 1* employ transformations, the following Proofs 2 and 3 depend on dissecting figures and rearranging the parts in clever ways.

The reference to dissections (decompositions) is quite natural as the measure "area" has the following properties:

1. Squares are used as units.
2. Congruent shapes have equal area.
   (Formally formulated: Area is invariant under rigid motions.)
3. If a polygon is dissected in disjunct parts the sum of the areas of the parts is equal to the area of the whole polygon.
   (Formally formulated: The area measure is additive.)

As a consequence of 1. and 2. equi-decomposable polygons have the same area. This relationship is also basic for the derivation of formulae for the areas of special polygons. So decomposition proofs are well embedded in the curriculum.

Obviously the following three proofs are the result of playing with shapes with the intention to get closed figures.

Proof 2 (*Indian Decomposition Proof*) This proof comes to us from the ancient Indians. It gives a direct solution of the problem to construct a square whose area is equal to the sum of the areas of two given squares.

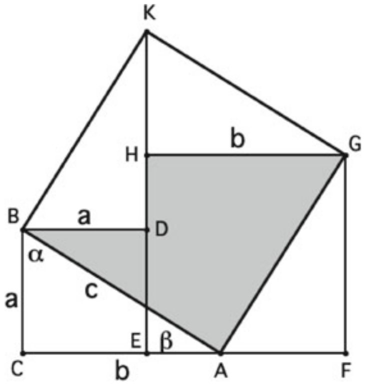


Fig. 9

Construction 1 (Fig. 9): Draw right triangle ABC with sides $a = BC$, $b = AC$, $c = AB$. Describe square $CEDB$ on side BC, extend CA and draw square $EFGH$ (side b). Extend EH such that $HK = a$ and draw quadrilateral $AGKB$.

Statement 1 The sum of the areas of squares $CEDB$ and $EFGH$ is equal to the area of square $AGKB$.

Proof Let α and β be the acute angles in the right triangle ABC. As the sum of angles in all triangles is 180° we have the basic (and frequently used!) relation $\alpha + \beta = 180° - 90° = 90°$.

By construction $AF = CE + EF - CA = a + b - b = a$ and $DK = EH + HK - ED = b + a - a = b$. Therefore all triangles ABC, GAF, GKH, and KBD have sides a, b subtending a right angle and so are congruent. As a consequence all sides of $AGKB$ have equal length c and all angles have measure $\alpha + \beta = 90°$, that is, $AGKB$ is a square.

The area c^2 of $AGKB$ is equal to the sum $a^2 + b^2$ as AGKB is composed of the shaded polygon and two triangles and the original squares are covered by the same polygon and two congruent triangles.

On Sect. 3.2 we will meet Fig. 19 which turns out as nothing but Fig. 9, rotated by 180°.

Proof 3 (*Geometric-Algebraic Proof*) This proof relates the Pythagorean theorem to the binomial formula $(a + b)^2 = a^2 + 2ab + b^2$, another fundamental topic of school mathematics.

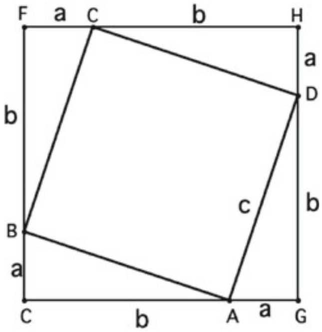


Fig. 10

Construction 2 (Fig. 10): Given lengths a, b we construct a square with side $a + b$ and inscribe a quadrilateral $ADCB$ which is a square (why?). As the area of each of the right triangles surrounding $ADCB$ is $\frac{1}{2} \cdot ab$ we get

$$c^2 = (a + b)^2 - 4 \cdot \tfrac{1}{2}ab = a^2 + 2ab + b^2 - 2ab = a^2 + b^2.$$

Exploration 8

Cut a square frame (side $a + b$) and four right triangles with legs a, b out of a piece of cardboard. The triangles can be put into the frame in different ways (see Figures 11a, 11b, and 11c).

1. Derive the Pythagorean theorem from Figures 11a and 11b and also from Figures 11c and 11b without using algebra. Compare these geometric proofs with proof 3.
2. Compare Figures 11b and 11c with Fig. 9 (proof 2). Can you extend Fig. 10 such that both Fig. 11b and Fig. 11c are visible in the extended figure?

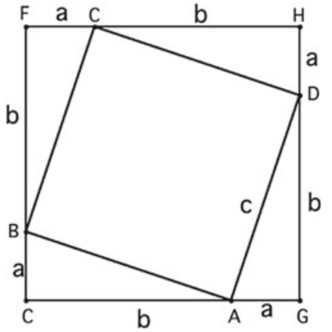
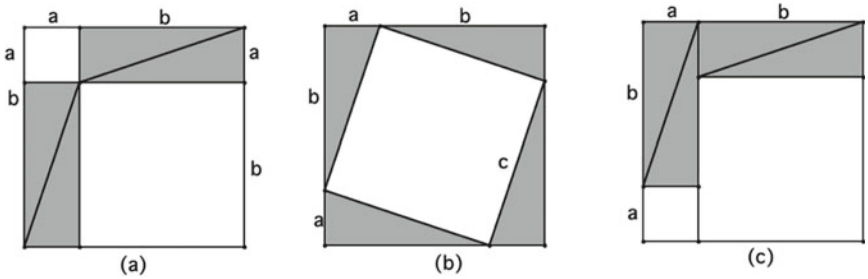


Fig. 10



(a) (b) (c)

Fig. 11

Proof 3* (*Bhaskara's Proof*) This proof is credited to the Hindu mathematician Bhaskara, who lived in the twelfth century, but it is much older and likely to have been known to the Chinese before the time of Christ.

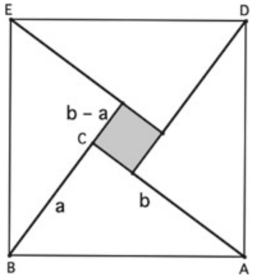


Fig. 12

The "Bhaskara" Fig. 12 arises from Fig. 10 (Proof 3) by folding the four right triangles inside the square. A careful check of lengths and angles reveals that the small quadrilateral inside is a square with side $b - a$. Therefore

$$c^2 = 4 \cdot \tfrac{1}{2}ab + (b - a)^2 = 2ab + b^2 - 2ba + a^2 = a^2 + b^2.$$

In this case there is no immediate purely geometric interpretation as before. However, we will come back to this problem later.

Proof 4 (*Similarity Proof*) It is an interesting question for historians which proof might have been given by Pythagoras himself. van der Waerden (1978) concludes from the context in which the Pythagoreans lived and worked that they might have used the self-similarity of a right triangle, that is its decomposability into two triangles similar to it. This proof runs as follows (see Fig. 13):

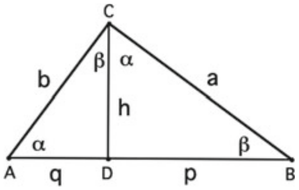


Fig. 13

The altitude dropped from vertex C divides the right triangle ABC into two right triangles with angles equal to the original triangle (why?). Therefore BCD and CAD are similar to ABC. This gives the proportions

$$\frac{p}{a} = \frac{a}{c}, \quad \frac{q}{c} = \frac{b}{c}$$

that can be transformed into

$$p = \frac{a^2}{c}, \quad q = \frac{b^2}{c}.$$

As $p + q = c$ we get $c = p + q = \frac{a^2}{c} + \frac{b^2}{c}$ and finally $c^2 = a^2 + b^2$.

Note that area doesn't play any role in this proof. The geometric basis is provided by proportions of lengths arising from similarity. The squares are the result of a purely algebraic manipulation of symbols standing for lengths. However, it is possible to interpret Fig. 13 in terms of area. This leads us to

Proof 4* (*Similarity/Area Proof*) Consider Fig. 13 once more. Triangles BCD and CAD are small copies of triangle ABC. Therefore the lengths of the sides of BCD and CAD can be obtained by reducing the lengths of the corresponding sides of ABC by the factor $\frac{a}{c}$ and respectively the factor $\frac{b}{c}$. So we have

$$\text{Area}\,(BCD) = \frac{a^2}{c^2} \cdot \text{Area}\,(ABC)$$
$$\text{Area}\,(CAD) = \frac{b^2}{c^2} \cdot \text{Area}\,(ABC).$$

As the sum of the areas of BCD and CAD is equal to the area of ABC we arrive at

$$\begin{aligned} \frac{a^2}{c^2} \cdot \text{Area}\,(ABC) + \frac{b^2}{c^2} \cdot \text{Area}\,(ABC) &= \text{Area}\,(ABC) \\ \left(\frac{a^2}{c^2} + \frac{b^2}{c^2}\right) \cdot \text{Area}\,(ABC) &= \text{Area}\,(ABC) \\ \frac{a^2}{c^2} + \frac{b^2}{c^2} &= 1 \\ a^2 + b^2 &= c^2. \end{aligned}$$

Note: If a dilatation with scale factor k is applied, areas are transformed by the *square* k^2. For example, area is multiplied by 4 if lengths are doubled, and multiplied by $\frac{1}{4}$ (that is, divided by 4) if lengths are halved.

Exploration 9

Compare Proofs 4 and 4*: In both proofs each of the two small triangles is first related to the big triangle separately. How? Then all three triangles are brought together. What is the crucial relation combining the three triangles and leading to the Pythagorean theorem in each proof? In other words: the Pythagorean theorem expresses an equality of areas. On what relationship is this equality based in each proof?

Reflective Problem 1

Analyze the proofs in this section: Where in the proof is the existence of a right angle crucial?
On which geometric or algebraic concepts is each of them based? Which geometric transformations are used? How do these affect area, length, measure of angles? Which algebraic formulae are used? Which step establishes the "equals" sign inherent in the theorem?
Evaluate the proofs: Which of them do you find easiest, which one most demanding? List them in order of increasing difficulty. Do you find them equally sound? If not, why? Which proof do you find most convincing, which one most interesting? Why? Do you prefer the algebraic or the geometric proofs?
Discuss your views with your fellow students; in particular compare your "difficulty" lists.

3.2 Heuristic Approaches to the Pythagorean Theorem

> We should orientate our teaching more on problems than on theories; a theory should be taught just as far as necessary for framing a certain class of problems.
>
> Giovanni Prodi

From the point of view of mathematical learning the mere study of proofs is not satisfactory, as it presents mathematics as a corpse laid down for an autopsy. Certainly logical analyses have their merit for recognizing conceptual relationships. However, in order to design teaching units that stimulate students to explore, describe, explain, and apply patterns we have to go back to the source of mathematical activity, that is, to *mathematical problems* inside and outside of mathematics. It is of central importance that students are offered the opportunity to experience mathematical concepts, theorems and techniques as *answers to problems* and as starting points for new problems. Otherwise it will be almost impossible for them to grasp the meaning of mathematics and to develop confidence in the use of it.

Our next task will be then to find appropriate problems that can lead to the discovery of the Pythagorean theorem and to explanations, that is proofs, of it.

The general direction of search is clear: We have to investigate situations in which the Pythagorean theorem is naturally used and examine if the context is strong enough in order to "generate" the theorem and to establish a proof.

Two approaches are offered below.

Approach 1 Clairaut's Approach (Clairaut 1741, sections 16,17)
A.C. Clairaut (1713–1765) was one of the most famous French mathematicians of the 18th century. He was a mathematical prodigy and wrote his first published mathematical paper on four spatial curves discovered by him as a twelve-year-old. Another paper of his attracted the attention of members of the French Academy

of Science who at first couldn't believe that a sixteen-year-old had written such an ingenious and profound paper of 127 pages. By special order of the King Clairaut was appointed a member of the Academy at the age of 18. It remained the only exception ever made to admit a person under 20 to the Academy.

Clairaut was also very much interested in teaching mathematics and as he strongly objected to the formalistic style of the textbooks used at his time, including Euclid's *Elements of Mathematics*, he set out to write books on elementary geometry and algebra in a quite different style. In the preface of his *Elémens de Géometrie* (Clairaut 1743) he explains his views on learning and teaching as follows:

> Although geometry is an abstract field of knowledge, nobody can deny that the difficulties facing beginners are mostly due to how geometry is taught in elementary textbooks. The books always start from a large number of definitions, postulates, axioms and some preliminary explanations that appear to the reader as nothing but dry stuff. The theorems coming first do not direct the students' mind to the interesting aspects of geometry at all, and, moreover, they are hard to understand. As a result the beginners are bored and rejected before they have got only the slightest idea of what they are expected to learn.
>
> In order to avoid this dullness attached to geometry some authors included applications in such a way that right after the theoretical treatment of the theorems their practical use is illustrated. However, in this way only the applicability of geometry is shown without facilitating the learning of it. As any theorem precedes its applications the mind is brought into contact with meaningful situations only after having taken great pains in learning the abstract concepts.
>
> Some thoughts on the origins of geometry made me hope to avoid these unpleasant difficulties and to take students' interests seriously into account. It occurred to me that geometry as well as other fields of study must have grown gradually; that the first steps were suggested by certain needs, and that these could hardly have been too high as it were beginners who made them for the first time. Fascinated by this idea I decided to go back to the possible places where geometric ideas might have been born and to try to develop the principles of geometry by means of a method natural enough to be accepted as possibly used by the first inventors. My only addendum was to avoid the erroneous attempts these people necessarily had to make.

Exploration 10

Compare Clairaut's view on problem-oriented teaching with statements on "Mathematics as Problem Solving" in the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989, pp. 7, 66, 75-77, 125, 137-139). What arguments are put forward in favor of problem-oriented teaching?

The problem chosen by Clairaut for introducing the Pythagorean theorem was this:

Determine the side c of a square whose area is the sum of the areas of two given squares with sides a and b.

In section 16 of his book he considers first the special case $a = b$ by asking how to construct a square whose area is twice the area of a given square.

The solution of this special case is fairly easy if one takes two copies of the given square, draws the diagonals, and rearranges the four triangles (see Fig. 14a and b). In concrete form four congruent isosceles triangles can be cut from cardboard and arranged in two ways corresponding to Fig. 14a and b ("square puzzle").

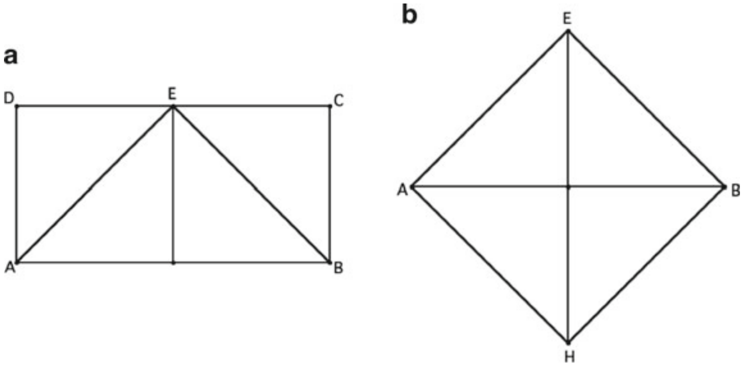


Fig. 14

In section 17 of his book Clairaut addresses the general case:
How to construct a square whose area is the sum of the areas of two different given squares?

The straightforward transfer from the special to the general case (see Fig. 15) is not successful, however, at least not immediately. Figure 16 does not "close".

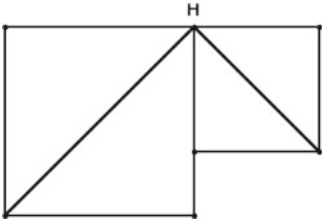


Fig. 15

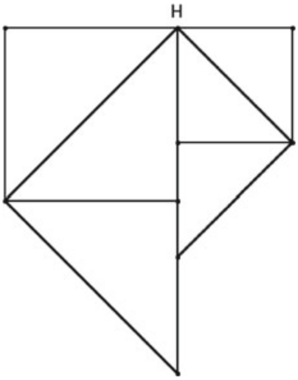


Fig. 16

But the construction can be adapted: If one dissects Fig. 16 by starting from a different point H (see Fig. 17) the new Fig. 18 is an “improvement.”

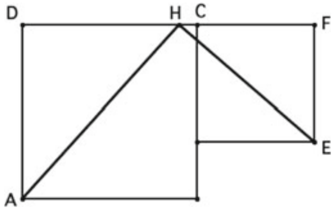


Fig. 17

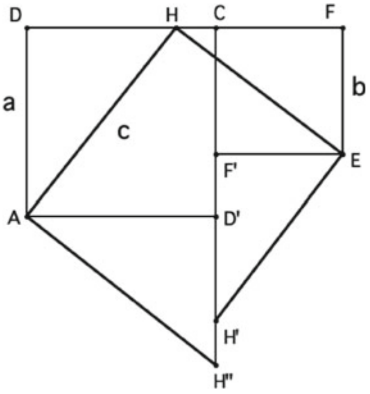


Fig. 18

Clairaut continues: “Following this idea it is quite natural to ask if it is possible to find a point H on DF such that

1. the triangles ADH and EFH if rotated around A resp E into the positions $AD'H'$ and $EF'H'$ meet in H',
2. the four sides AH, HE, EH' and $H'A$ are equal and form right angles.

It is easy to see that H is determined by $DH = CF$ $(= b)$ or $HF = DC$ $(= a)$. (See Fig. 19).”

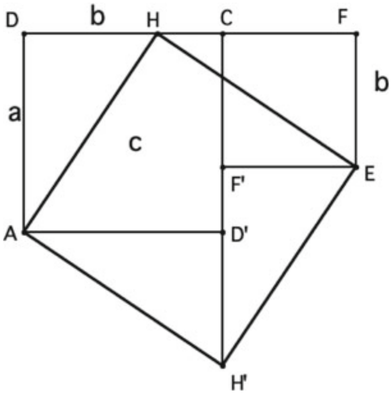


Fig. 19

The problem is now solved and all that remains to do is to introduce the sides a, b, c and to state that by construction $c^2 = a^2 + b^2$. The figure is determined by the right triangle AHD and can be drawn by starting from an arbitrary right triangle AHD. Therefore $c^2 = a^2 + b^2$ holds for the sides of *any* right triangle.

Figure 19 is well known to us: It is nothing but Fig. 9 of the "Indian decomposition proof" (Proof 2). While this figure came out of the blue in Sect. 1 it appears here *within the solution of a problem*, and the Pythagorean theorem gives the answer to this problem. We have in this example a good illustration for the difference between a proof embedded solely into a net of logical relationships and a proof embedded into a meaningful context.

Exploration 11

Use the software The *Geometer's Sketchpad* or *Geogebra* for representing Clairaut's approach in a dynamic way.

Special case: First draw figure 14a. Rotate AED around A by 270° and triangle BCE around B by $-270°$ (or 90°). You get a combination of Figures 14a and 14b.

General case: Draw Figure 17 starting with segment DF and choose H as a (moving) point on DF. Rotate AHD around A by 270° and EFH around E by $-270°$ (or 90°). You get Fig. 18. By moving H on segment DF points H' and H'' move on line CD', and you can easily find the position of H when the figure "closes" (see Fig. 19).

Approach 2 The Diagonal of a Rectangle

Our second approach starts from the following problem:
How long is the diagonal of a rectangle with sides a and b?

This problem is interesting from the mathematical point of view, but it has also a reasonable real interpretation: *A rectangular frame with sides a, b is to be stabilized by means of a diagonal lath. How long should the lath be (see Fig. 20)?*

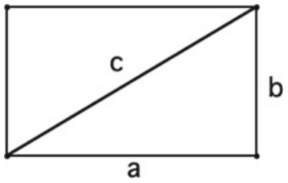


Fig. 20

If the Pythagorean theorem is known the answer is obvious: $c = \sqrt{a^2 + b^2}$. However, our aim is again to use this problem for "generating" the Pythagorean theorem.

How can we approach this problem? For example, we can compare a, b and c and find that c is longer than both a and b and smaller than $a + b$. We also can draw rectangles of different shapes, measure c, and establish a table.

a in cm	10	8	4	8	7.5	9
b in cm	5	5	3	6	7.5	7.5
c in cm	11.2	9.4	5	10	10.6	11.7

But how to *calculate* c? The heuristic strategy "Specializing" used by Clairaut is a reasonable strategy here, too. So let us consider first the special case of a square (see Fig. 21).

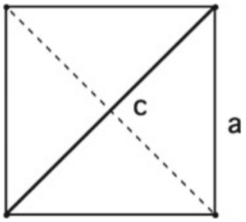


Fig. 21

How is the diagonal c of a square related to its side a?

Reflective Problem 2

Think about this problem. Note that one diagonal divides the square into two congruent right triangles with hypotenuse c and altitude $c/2$. So there are two ways of calculating the area that can be used to derive the relationships $c^2 = 2a^2$ and $c = \sqrt{2} \cdot a$.

$c^2 = 2a^2$ "cries" for a geometric interpretation. It is provided by the "square puzzle" from approach 1: Four congruent right isoceles triangles can be put together to form either one big square or two small squares (see Fig. 22a and b).

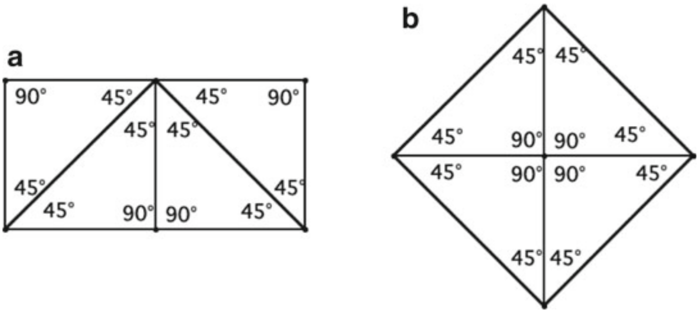


Fig. 22

As before we try to generalize this result to rectangles, that is, we look for a generalized "puzzle" establishing the Pythagorean theorem for arbitrary right triangles.

Is it possible to recombine the four halves of two congruent rectangles to make a square whose side is the diagonal of the rectangle?

Exploration 12

Cut four congruent right triangles from cardboard (see Fig. 23) and think about this problem first for yourself. Can you make a square shape with side *c* with these pieces?

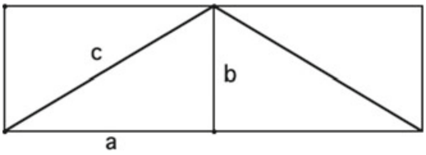


Fig. 23

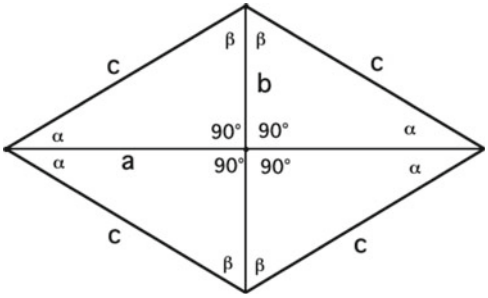


Fig. 24

A first attempt leads to Fig. 24 which, however, is not a square, but only a rhombus: All sides are equal, but the angles are different—two of them are 2α and two of them are 2β.

However, because of the basic relation $\alpha + \beta = 90°$ we could try to combine the four right triangles in a slightly different way (see Fig. 25).

We arrive at three equal sides, two right angles, and an isolated right triangle. The question is:
Does the fourth triangle really fit in? The dotted line indicates a square "hole" with side $a - b$. Because of $a - (a - b) = b$ and $b + (a - b) = a$ the gap is exactly filled indeed by the fourth triangle. So we get a square with side c but, alas, with a square "hole" inside (see Fig. 26).

That the angles of the "hole" are right angles follows from the right angles of the triangles.

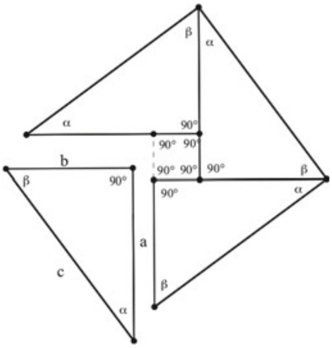


Fig. 25

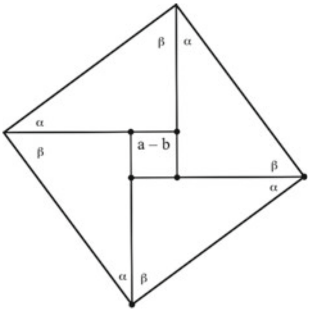


Fig. 26

Nevertheless we can calculate c:

$$c^2 = 4 \cdot \frac{1}{2}ab + (a-b)^2$$

$$c^2 = 2ab + a^2 - 2ab + b^2$$

$$c^2 = a^2 + b^2$$

$$c = \sqrt{a^2 + b^2}$$

This is the formula we were looking for: The side c is expressed as a function of a and b.

Again, Fig. 26 is well known to us: It is exactly Fig. 12 used by Bhaskara (Proof 3*). In marked contrast to that presentation, the figure appears here within the solution of a problem. So we have another illustration of the difference between a formal proof within a deductive structure and an informal proof arising from a meaningful context.

As in the special case we want to understand $c^2 = a^2 + b^2$ in purely geometric terms.

The square with side c can be formed by means of a puzzle consisting of five pieces ("Bhaskara-Puzzle"): four congruent rectangular pieces with sides a, b and a square piece with side $a - b$. Can these five pieces be recombined to form a shape composed of a square with side a and a square with side b?

Exploration 13

Cut the five pieces of the "Bhaskara-Puzzle" from cardboard and show geometrically that $c^2 = a^2 + b^2$. You have to arrange the five pieces such that they cover the union of a square with side a and of a square with side b. Hint: Fig. 9 or Fig. 19.

Exploration 14

Reexamine the logical line in approaches 1 and 2: At what places is the assumption of right angles crucial?

3.3 Exploring Students' Understanding of Area and Similarity

Concepts are the backbone of our cognitive structures. But in everyday matters concepts are not considered as a teaching subject. Though children learn what is a chair, what is food, what is health, they are not taught the *concepts* of chair, food, health. Mathematics is no different. Children learn what is number, what are circles, what is adding, what is plotting a graph. They grasp them as *mental objects* and carry them out as *mental activities*. It is a fact that the concepts of number and circle, of adding and graphing are susceptible to

> more precision and clarity than those of chair, food, and health. Is this the reason why the protagonists of concept attainment prefer to teach the number concept rather than number, and, in general, concepts rather than mental objects and activities? Whatever the reason may be, it is an example of what I called the anti-didactical inversion.
>
> Hans Freudenthal

The mathematical and heuristic structure of subject matter form only two of the three strands that have to be twisted in the design of teaching. The third equally important one is knowledge of the students' cognitive structures as far as they are relevant for the topic to be learned.

Our mathematical analyses have shown that the Pythagorean theorem is fundamentally related to the concepts of area and similarity. Therefore it is necessary to provide data on the psychological development of these concepts. We cannot give a systematic and coherent review of research here. Instead we concentrate on a few interesting studies that give a first orientation and—what is even more important—also provide a basis for doing similar studies. The central part of this section is "Clinical interviews on area and on doubling a square," where the reader is stimulated to do some study of his or her own.

The basic message of this chapter is this: Mathematical concepts are neither innate nor readily acquired through experience and teaching. Instead the learner has to reconstruct them in a continued social process where primitive and only partly effective cognitive structures that are checkered with misconceptions and errors gradually develop into more differentiated, articulated and coordinated structures that are better and better adapted to solving problems. For teachers this message is of paramount importance: Concepts must not be presupposed as trivially available in students nor as readily transferable from teacher to student. On the contrary, the teacher must be prepared that students often will misunderstand or not understand what he or she is talking about. To have a feeling for students' misconceptions, to be able to dig into students' thinking until some solid ground appears that may serve as a basis for helping the students to reconstruct their conceptual structures on a higher level, to interact with students particularly in seemingly hopeless situations—that is the supreme mark and criterion of a competent teacher.

Doubling a Square: Plato's dialogue Meno

The Greek philosopher Plato (ca. 429–348 B.C.) is an important figure for mathematics education as in his philosophic system mathematics played a fundamental role. Relevant for teaching and learning and therefore frequently referred to is his dialogue *Meno* that centers around the fundamental questions if virtue can be taught and where knowledge does come from.

One part of this dialogue is particularly interesting as perhaps the oldest recorded lesson in mathematics: Socrates teaches, or better interviews, a boy on how to double a square (Plato 1949).

The structure of the interview is as follows:

1. A 2 × 2-square is presented and the boy is asked to find a square of double size (see Fig. 27)

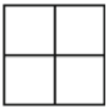


Fig. 27

2. Although the boy predicts the area of this new square as 8 square feet, nevertheless his first suggestion is to double the sides. This leads to the 4 × 4-square (see Fig. 28) that turns out as four times as big instead as twice as big—a cognitive conflict for the boy!

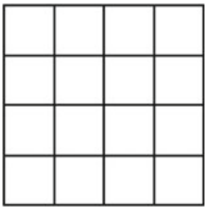


Fig. 28

3. In order to correct his mistake the boy offers the 3 × 3-square (lying between the 2 × 2 and the 4 × 4-square) as the solution (see Fig. 29). Again Socrates arouses a cognitive conflict by having the boy calculate its area: to his own surprise the boy finds 9 square feet instead of the expected 8!

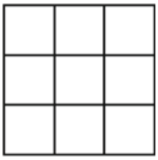


Fig. 29

4. Finally, it is Socrates who returns to the 4 × 4-square (see Fig. 30), introduces the diagonals and guides the boy to discover that the square formed by the diagonals has the required area of 8 square feet and therefore is the solution of the problem.

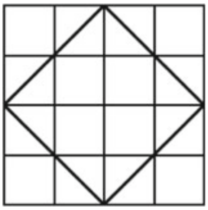


Fig. 30

Clinical Interviews on Area and on Doubling a Square

Plato's dialogue is interesting in our context not only because it deals with a special case of the Pythagorean theorem, the doubling of a square, but also because it can be considered as the ancient version of a psychological method that was fully developed by the great Swiss psychologist and epistemologist Jean Piaget (1896–1980) in the thirties and is now widely used in research: the "clinical interview" However, there is a basic difference in the views of Plato/Socrates and Piaget as far as the origin of knowledge is concerned. These Greek philosophers believed that knowledge was already *innate* in human beings. So they compared the teacher's task to that of a midwife: With a definite goal in mind the teacher has to make the student "remember" and bring his knowledge to light. In sharp contrast with this view Piaget sees knowledge not as something pre-fabricated inside or outside the learner but as the continued personal construction and reconstruction of the learner while interacting with and trying to adapt to the natural and social environment. Therefore the Piagetian interviewer unlike Socrates in Plato's dialogue is anxious *not* to guide the student to a definite end. The aim of the clinical interview is to uncover the student's authentic mental structures, not to subject him or her to any kind of "teaching". Therefore the interviewer must be open to what the student has to offer, try as far as possible to put him- or herself in the student's place and make sense of the student's thinking—also in case of strange and contradictory answers. He or she must not be content with just listening to students, but has to stimulate them to express their mental processes with words or actions, always following their fugitive thoughts. Questions like "How do you know?" or "Why do you think so?" and cautious counter-arguments for arousing cognitive conflicts like "But some other child told me ..." are essential elements of clinical interviews. In short, the clinical interview is a kind of "mental auscultation" analogous to the physical auscultation in medical checkups. For this reason it was called *clinical*.

It is important for student teachers to realize that the clinical method is valuable not only from the psychological but also from the pedagogical point of view: In conducting clinical interviews the student teacher acquires insight into children's thinking and becomes familiar with essential habits of good teachers—introducing children into a mathematical situation with parsimonious means and with clear explanations, showing interest for what they are doing, observing them without interrupting, listening to them, accepting their intellectual level, giving them time to work and to think, stimulating their thinking by sensitive questions and hints, and so forth (Wittmann 1985).

Of course, the clinical method is also extensively used in Piaget/Inhelder/Szeminska (1964), one of the major studies of children's geometric thinking. The book contains a chapter on doubling area and volume (Chap. XIII).

The following study done by student teachers was inspired by both Plato and Piaget. It may give a feeling for both Piaget-like studies into students' thinking and the clinical interview as a research method.

The following setting was used:

Material: 16 congruent squares (3 cm × 3 cm) and 32 triangles (half of one 3 × 3 square), made of cardboard.

Technique: Students were interviewed individually according to the following scheme:

1. Involve the student in an informal chat as a kind of warm-up.
2. Show the student the geometric forms and ask: *Which different figures can you build with these?*
3. After phase 2 is finished take four squares, form a 2 × 2-square, tell the student the following story, and conduct some "mental auscultation" on his or her understanding of area:
   Imagine that this is a pasture surrounded by a fence. It is just big enough to give grass for exactly eight cows. Now the farmer buys eight more cows and wants to fence off a pasture that is twice as big. As he likes squares the bigger pasture should be a square as well.
   Can you help the farmer and make a square of double size?

The above scheme is "half-standardized" in the sense that only some key informations and questions are prescribed that have to be reproduced in all interviews and so form the common core. All other interactions depend on the student's responses.

The following two interviews with the 11 year-old Dirk and with the fifteen-year-old Stefan give an impression of the wide range of students' abilities and thinking.

Dirk

1. While playing around with the forms and laying out a variety of figures Dirk explicitly states that four triangles can be arranged to make a larger triangle (see Fig. 31).

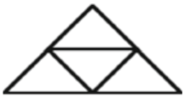


Fig. 31

2. In order to solve the pasture problem Dirk adds four squares and produces Fig. 32:

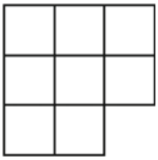


Fig. 32

Dirk: Oh, no, that doesn't work. That gives me nine squares, but I need eight.

Next he tries to attach a triangle to a square (see Fig. 33a).

When seeing that this is possible he builds Fig. 33b and adds another triangle (see Fig. 33c)

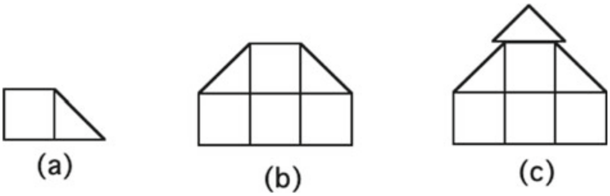


Fig. 33

Dirk: Oh, no, that doesn't fit!

His next figure is Fig. 34.

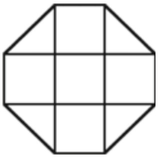


Fig. 34

He counts the squares: These are only seven squares. Must all sides be equal?

Interviewer: Yes. Do you think you can do it?

Dirk's next step is Fig. 35, a figure with an area equivalent to eight squares, but not a square.

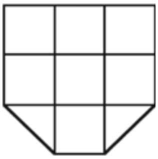


Fig. 35

Dirk, after thinking for a while: I could try it with triangles.

He first makes the original square (see Fig. 36a)

Then he supplements eight more triangles (see Fig. 36b).

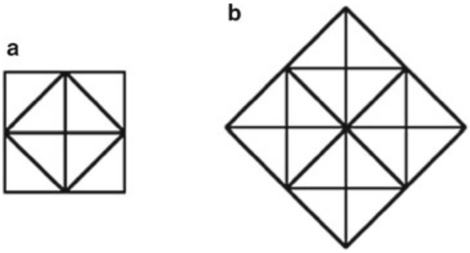


Fig. 36

Dirk: I think, that's it!

Interviewer: Why do you think it is twice as big as the old square?

Dirk: Inside you have the old square with eight triangles, and in addition eight new triangles.

Interviewer: How did you hit upon the idea to arrange the triangles this way?

Dirk: I saw that two triangles make a larger triangle (see Fig. 37) that can be added to the square.

Fig. 37

Stefan

1. Stefan builds only few figures, for example a rectangle and a "house" (see Fig. 38a and b).

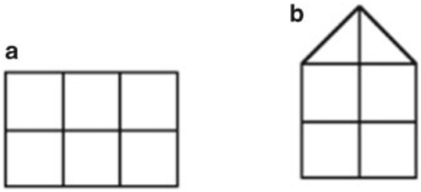


Fig. 38

2. Stefan immediately lays a 3 × 3-square (see Fig. 39)

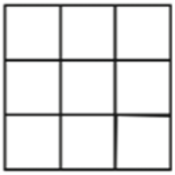


Fig. 39

Interviewer: Do you think this is right for twice as many cows? Is it really double the size?
Stefan: Yes.
Interviewer: How do you know that?
Stefan: These are four [*he points to the original square*] and these are ... five—there is room for more cows.
Interviewer: Can you also build a pasture for exactly sixteen cows?
Stefan: I do not see how.
He adds two squares to the original square (Fig. 40).

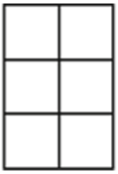


Fig. 40

Stefan: No. In this way I again get one more ... It is not possible.
Interviewer: What about using these triangles?
Stefan: No. Two triangles make a square again.
Interviewer: So you think, it's not possible?
Stefan: No, it's not possible. Either you have to take one square more or you must build a rectangle.

Exploration 15

Analyze Figures 33c, 34 and 35 in the interview with Dirk. How far are they away from the solution? Keep in mind that the solution has to meet two requirements: it has to be a square *and* it has to have an area of 8 unit squares. Fig. 34 can be extended to a square in two ways: 1. By adding four given triangles to the four longer sides of the octagon one gets a 3 × 3 square. 2. But compare Fig. 33c with Fig. 34. Obviously Dirk tries to add triangles to the smaller sides of the octagon. The given triangles are too big, as Dirk recognizes: What triangles would be necessary to extend the octagon to a square in this second way? How many unit squares would this second square have?

Do a similar analysis with the second interview. What is the biggest block in Stefan's thinking?

The main findings of some thirty interviews with students in the age range eleven to fifteen were the following:

1. Only a few students built first the 4 × 4-square and they were aware *at once* that it was too big, that is, the first misconception in Plato's dialogue was not observed.
2. However, almost all students arrived at the 3 × 3-square somewhere in the interview either taking it for the solution (as the boy in Plato's dialogue) or using it as a step towards the solution.
3. Students that flexibly operated with forms (like Dirk) had a much higher chance of finding the solution than students who were "fixed" to certain tracks (like Stefan). The variable "age" was of minor importance.

Reflective Problem 3

Form pairs or triples of student teachers and conduct clinical interviews with secondary students on doubling a square by using the material and the technique described above. One of you should conduct the interview, the other one or two should act as observer(s) and take written notes or serve the cassette recorder or the video camera. Don't forget to change roles.

In retrospect you can also examine how far you have fulfilled the requirements for clinical interviews listed at the beginning of this subsection or in Ginsburg 1983.

Exploration 16

Show secondary students the "Bhaskara-puzzle" (Exploration 13) and ask them to form "different figures". If they do not hit upon the square by themselves ask them to make one.

Student Conceptions and Misconceptions about Area Measure and Similarity

Piaget's research on area formed the starting point for many investigations that share the following general framework: The student is offered a series of items where figures are to be constructed and transformed in various ways—cut from paper, moved around, reflected, decomposed, rearranged, enlarged and reduced. The student is always asked to describe, predict and explain how area "behaves" under these transformations. The answers indicate how well he or she has attained the concept of area.

We will see in section called "The Operative Prinziple" that in Piaget's theory of cognitive development the flexible use of "operations" is considered as the cornerstone of intelligent behavior in mathematics and beyond. Of course the nature of operations differs from domain to domain: in arithmetic we deal with number operations, in calculus we use transformations of functions, in combinatorics we operate with combinatorial formulae, and so forth. Nevertheless in all these fields there is an operative structure of knowledge. The following analysis of operations connected to area and similarity has therefore far-reaching importance as an instructive special case.

The present subsection tries to give an idea of research that clearly follows the Piagetian paradigm and covers the age range from eight to fifteen. The aim is just to establish a feeling for area as a concept *developing in students' minds*.

In clinical interviews with eight- to eleven-year-olds Wagman (1975) used the following tasks that directly reflect the basic properties of the concept of area: 1. Use of squares as units, 2. Invariance under rigid motions, 3. Additivity.

Unit Area Task. The investigator presents three polygons that can be covered by unit squares and asks the child to find out how many squares are needed in each case. The necessary squares are piled besides each polygon. In the second part the child is given a large number of triangular tiles each of which is equal to one half the square tile. The child is asked to find out how many of these triangular tiles are needed to cover the polygons.
Congruence Axiom Task: The investigator presents the child with two congruent isosceles right triangles, one blue, the other one green. The child is asked how many white triangles (of half linear dimensions) are necessary for tiling the blue triangle. After discovering the answer (4) the child is asked to guess without trying how many white triangles will be needed to tile the green triangle.

Additivity-of-Area Tasks

1. The child is presented with two polygonal regions a. with equal areas, b. with different areas. Given a set of smaller shapes the child is asked to cover each of the two polygons and to decide if they have the same or different amounts of space.
2. Polygons are decomposed and the pieces are rearranged to form another polygon. The child is asked to compare the areas.

In all these tasks the investigator encourages the child to work on the materials and to give explanations: "How do you know?", "Why do you know?", "Are you sure?".

As we shall see, these tasks test the children's ability to operate flexibly and effectively with shapes and their understanding of the concept of area.

In her study with 75 children from eight to eleven years, Wagman found that 6 of them were still in a “pre-measurement” stage, 31 showed some first understanding of area, 35 could use all properties of the area concept in simple cases, and only 4 displayed full mastery (Wagman 1975, 107).

In a study with large numbers of secondary students (twelve to fourteen years) Hart (1981, 14-16) used the following tasks that are similar to some of Wagman’s tasks and also test the mastery of the properties *invariance* and *additivity*:

1. A machine makes holes in two equal squares of tin in two different ways (see Fig. 41 A, B). Students are asked to compare the amount of tin in A and B.

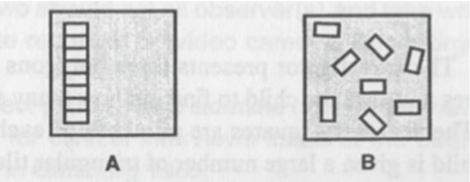


Fig. 41

2. A square A is cut into three pieces and the pieces are arranged to make a new shape B (see Fig. 42). Students are asked to compare the areas of A and B.

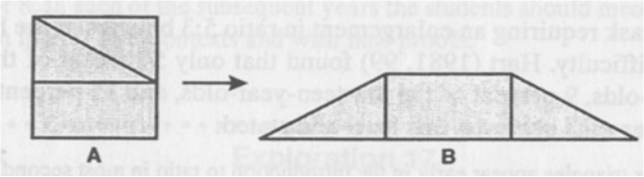


Fig. 42

The result revealed that about 72 percent of the total population could successfully answer both questions.There were no major differences between the age groups.

In comparing Wagman’s findings with these results we recognize a substantial increase in understanding the concept of area for the majority of students. However, the concept of area is by no means mastered by all secondary students.

In the past decade research on the development of the similarity concept has been intensified. A typical problem used in the International Studies of Mathematical Achievement in 1964 and in 1982 for eight graders is the following one:

On level ground, a boy 5 units tall casts a shadow 3 units long. At the same time a nearby telephone pole 45 units high casts a shadow the length of which, in the same units, is

A. 24
B. 27
C. 30
D. 60
E. 75

The results are remarkable: 56 percent of the students chose the correct answer (27) at the end of the school year 1963/64, whereas only 41 percent did so at the end of the school year 1981/82.

In examining students' thinking on ratio and proportion Hart (1981, 97–101) used a battery of items among them the following task:

The students are shown a 3 cm × 2 cm rectangle (Fig. 43) and a base line of 5 cm (Fig. 44) that is to be completed "so it is the same shape but larger" than the given rectangle.

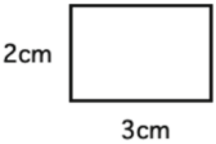


Fig. 43

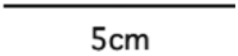


Fig. 44

This task requiring an enlargement in ratio 5 : 3 belongs to the highest level of difficulty. Hart (1981, 99) found that only 5 percent of the thirteen-year-olds, 9 percent of the fourteen-year-olds and, 15 percent of the fifteen-year-olds achieved this level and stated:

> Similar triangles appear early in the introduction to ratio in most secondary textbooks and children are expected to recognize when triangles are similar to each other and the properties they possess. On interview it was found that the word "similar" had little meaning for many children. In everyday language the word is used in a non-technical sense to mean "approximately the same". There was particular difficulty with the word when similar triangles or rectangles were under discussion. The test items dealing with similar figures (not triangles or rectangles) were some of the hardest on the test paper. Using ratio to share amounts between people "so that it is fair" seemed to be much easier than dealing with a comparison of two figures. In enlarging figures there is the danger of being so engrossed in the method to be used that the child ignores the fact that the resulting enlargement should be the same shape as the original ... The introduction of non-whole numbers into a problem does not make the question a little harder but a lot harder.

These findings are confirmed by a series of other papers (see, for example, the review by Lappan and Even 1988).
Summary:
Psychological research on the concepts of area and similarity suggests the following conclusions for teaching the Pythagorean theorem:

1. A satisfactory treatment of the Pythagorean theorem can only be reached within a long-term perspective of the curriculum. For coming to grips with the concepts of congruent and similar figures as well as of linear and area measure students need rich opportunities to operate with figures within meaningful contexts. Work has to start in the early grades with concrete materials, it has to be continued with concrete materials and drawings in the middle grades, and should gradually be extended to more symbolic settings in higher grades. It is only in this way that students can understand the properties of area and relationships between figures based on area and exploit them with mental flexibility for solving problems as well as for proving theorems.
2. The approach to the Pythagorean theorem via similarity is conceptually much more difficult for students than the approaches via area preserving dissections and recombinations of figures. Therefore similarity is not appropriate for introducing the Pythagorean theorem. However, it is a good context for taking up the Pythagorean theorem at a more advanced level. Because of the fundamental importance of the Pythagorean theorem the first encounter with this theorem should take place at latest in grade 7 or 8. In each of the subsequent years the students should meet the theorem in ever new contexts and with new proofs.

Exploration 17

The psychological findings on childrens' thinking as illustrated in the present section show that students of a given age range differ considerably in their understanding of basic concepts. The teacher is therefore confronted with the following fundamental problem: How to cope with this wide spectrum of student abilities?
List the major measures that are recommended to teachers in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) for addressing this problem.

4 Designing Teaching Units on the Pythagorean Theorem

The mathematical and psychological analyses and activities in the preceding sections have set the scene for attacking the central problem of this paper: the design of teaching units on the Phytagorean theorem.

From what has been said so far in this paper the following boundary conditions for designing introductory teaching units on the Pythagorean theorem should be clear:

1. Students should be faced with a *problem* that is typical for the use of the Pythagorean theorem and rich enough to derive and explain (prove) the theorem.
2. The conceptual underpinning of the unit should be as firmly rooted in students' basic knowledge as possible.
3. The setting should be as concrete as possible in order to account for different levels in the mastery of basic concepts, to stimulate students' ideas and to facilitate checking.

It is typical for all kinds of design that the designer cannot derive his "products" by means of logical chains of arguments from a scientific basis. Instead he or she has to invent them relying on his or her imagination and using the scientific basis for checks of validity, reliability and effectiveness. Therefore the above boundary conditions do not determine one teaching unit but leave room for the designer's preferences. It is important to keep this in mind and to interpret the following units as suggestions, not as dogmatic prescriptions.

Exploration 18

Before analyzing the following teaching units resume Exploration 6 and do some brainstorming on ideas how to introduce the Pythagorean theorem. Which mathematical or real problem situations do you think appropriate at which school level? What approaches are chosen in textbooks?

Two introductory teaching units on the Pythagorean theorem are offered below. One of them is based on ideas developed in the section on heuristic approaches (pages 15–24), the other one is taken from a Japanese source and puts strong emphasis on technology.

4.1 Approaching the Pythagorean Theorem via the Diagonal of a Rectangle

The problem of determining the diagonal in a rectangle with sides a, b ($a \geq b$) seems to be an appropriate context for introducing the Pythagorean theorem in grade 8. Students can explore this problem by first measuring diagonals, then considering the special case $a = b$, and finally trying to transfer the idea from this case to the general case. The solution and the proof depend on dissecting and recombining figures. These area-preserving operations can easily be illustrated by using two puzzles made of cardboard:

1. The "square puzzle" consisting of four semi-squares with side a.
2. The "Bhaskara puzzle" consisting of four right triangles (semi-rectangles) with legs a, b and a square of side $a - b$ (where $a > b$).

The language of puzzles is very powerful and allows for expressing Proof 3 of the Pythagorean theorem in the following way that seems to be a good orientation for work with students (for other approaches to the Pythagorean theorem using puzzles see Eaves 1953; Spaulding 1974; Engle 1976 and Beamer 1989). We take an arbitrary square piece of cardboard with side a and cut it along one of its diagonals (length c) into two isoceles right triangles (see Fig. 45).

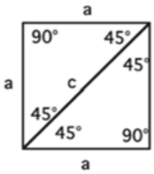


Fig. 45

We repeat this process with a second congruent square piece of cardboard and arrive at four isoceles triangles with legs a and acute angles of 45° (see Fig. 46). These triangles are pairwise congruent as the right angles and the legs are equal. The four pieces can be recombined to make a square with side c as the four right angles form one full angle at the midpoint and adjacent legs of lengths a fit together.

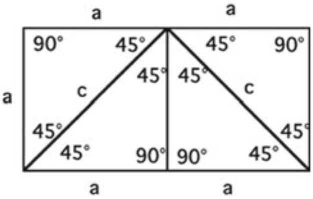


Fig. 46

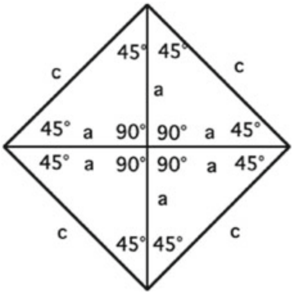


Fig. 47

All angles at the four vertices are right angles as $45° + 45° = 90°$. Figure 47 is really a square.

Obviously, the area of the square with side c is equal to the sum of the areas of two squares with side a: $c^2 = 2a^2$.

In a similar way we dissect two congruent rectangular pieces of cardboard with sides a and b ($a > b$) into four right triangles with legs a and b (see Fig. 48).

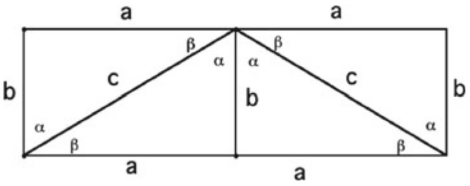


Fig. 48

All four triangles are congruent as they coincide in the right angles and the legs a and b. The sum of the acute angles α and β is 90° (= 180° − 90°).

The four triangles can be recombined to make a square with a small square hole. At each corner the triangles fit perfectly as $\alpha + \beta = 90°$ (see Fig. 49).

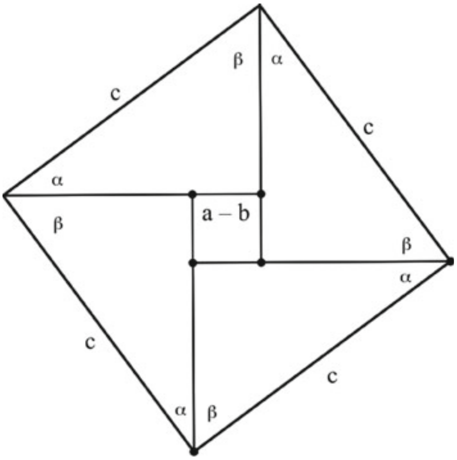


Fig. 49

The square inside has four right angles and equal sides of length $a - b$. Hence it is a square.

We fill the hole by a square piece of cardboard and recombine the five pieces as in Fig. 50.

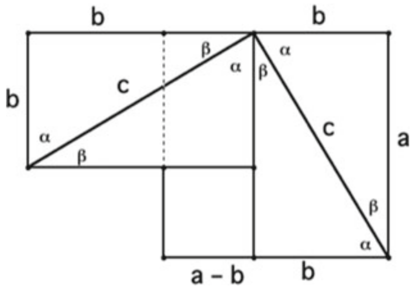


Fig. 50

The resulting figure consists of two rectangles with sides a, b and a square with side $a - b$. The dotted line decomposes the figure into two quadrilaterals: As $(a - b) + b = a$, $a - (a - b) = b$ and all angles are right angles, the quadrilaterals are squares with sides a and b. Now the square with side c is composed of the same five pieces as the two squares with sides a and b. Therefore we have proved that $c^2 = a^2 + b^2$.

This description may sound a little clumsy, but it describes a procedure students can perform and comment *orally* quite easily, and this procedure explains *why* the relationship $c^2 = a^2 + b^2$ must be true: the line of arguments is a sound proof in an informal setting centered around the solution of a problem.

It appears as instructive to round out the unit by comparing the measurements in the table with the values calculated by means of the formula. Also the heuristic use of the Pythagorean theorem should be derived from this special context: given the lengths of two sides of a right triangle the length of the third side can be calculated.

As a result we arrive at the following plan for a teaching unit. The plan is presented in a "half-standardized" way directly analogous to the scheme used in conducting clinical interviews (see Sect. 3.3). The unit is divided into "episodes". At the beginning of each episode the teacher has to take the initiative. His or her crucial interventions (and only these!) are explicitly described. The further moves to be taken depend on students' ideas and therefore they have to be left open.

Teaching Plan

1. ***Presenting the guiding problem***
   Rectangles of different shapes are drawn on the blackboard (Fig. 51).

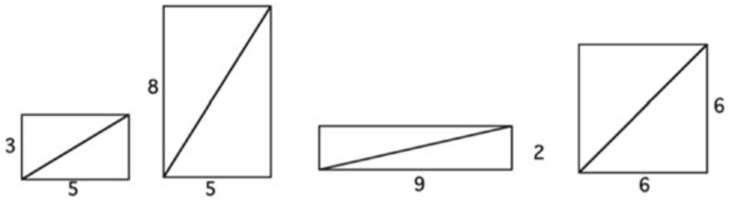
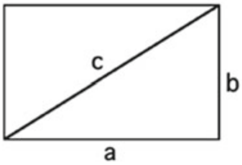


Fig. 51

The teacher explains the problem of finding the length of the diagonal. As an example making a lath for stabilizing a rectangular frame is mentioned.

It is only natural that the students will also suggest to measure the diagonals. The teacher recommends to draw a variety of rectangles and to measure the diagonals, and fixing the results in a table (Fig. 52).



a	5	5		
b	3	8		
c				

Fig. 52

At the end of this episode some data are collected in a common table on the blackboard.

2. ***Redefining the problem***
   The teacher redefines the problem as the typically mathematical problem of finding a formula for computing the diagonal c from the sides a and b. The advantage of a formula should be plausible to students.
   Students are stimulated to guess what such a formula could be like. The suggested ideas are written up and tested against the values in the table.
   At the end of this episode the students are informed about the steps to follow: Receiving some hints from the teacher they should try to discover and prove the formula as far as possible by themselves.
3. ***Specializing the problem: Diagonal of a square***
   Material: Congruent paper squares. As a first hint the teacher suggests to study squares as an easier special case.
   Each student gets some congruent paper squares and diagonalizes them. The task is to find an arrangement of squares such that a relationship between diagonal c and side a can be deduced.
   Figure 53 is almost inevitable and leads to the relationship $c^2 = 2a^2$, from which $c = \sqrt{2} \cdot a$ can be derived.

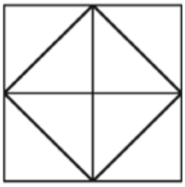


Fig. 53

The episode is concluded by a guided informal proof of the relationship $c^2 = 2a^2$ based on the transformation in Fig. 54.

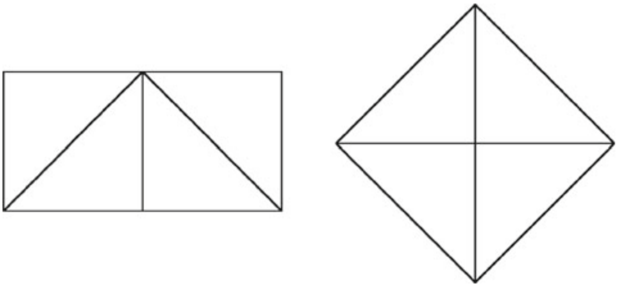


Fig. 54

4. ***Generalizing the solution: Diagonal of a rectangle***
   Material: Congruent paper rectangles. The teacher suggests to adapt the solution from squares to rectangles.
   Each student gets two paper rectangles, diagonalizes them and tries to make a square. Students are guided to discover the Bhaskara solution and to give an informal proof of the Pythagorean theorem (Fig. 55).

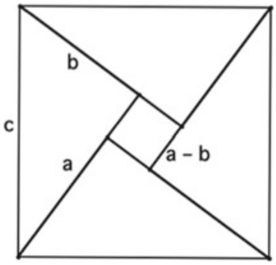


Fig. 55

$$c^2 = a \cdot \frac{ab}{2} + (a - b)^2 = a^2 + b^2$$

$$c = \sqrt{a^2 + b^2}$$

5. ***Discussing the formula***
   The teacher informs about the history of the Pythagorean theorem and about its importance. Students check the formula by comparing the measured values (episode 1) with the values obtained from the formula.

4.2 Japanese Approach to the Pythagorean Theorem

The Japanese volume *Mathematics Education and Personal Computers* contains a case study on the Pythagorean theorem as an example for improving the traditional format of teaching (Okamori 1989, 155-161). Instead of treating the whole class as one body the class was split up into small groups (four or five students) according to interests, academic abilities and social relationships. The idea was to offer students different approaches to the subject matter that might better serve the individual needs and preferences.

Each group was provided with a microcomputer that had been fed with an interactive software allowing for three different contexts to investigate and prove the Pythagorean theorem:

1. "Geometric-algebraic": The screen shows squares and dissections as presented in Proofs 3 and 3* (see Figs. 10 and 11a).
2. "Euclid dynamized": The screen shows a movie according to Proof 1* (see Fig. 8).
3. "Experimental": The screen shows a right triangle and the squares described on its sides. The medium size square is dissected according to Fig. 73 (see the dissection proof derived from problem 2 of Exploration 3 on page 4).

The following teaching plan shows a structure that is typical for Japanese mathematics education:

- The objectives are clearly defined.
- The steps are precisely described.
- Materials for students are carefully provided.
- At the end of the lesson the teacher summarizes what has been learned.

Teaching plan

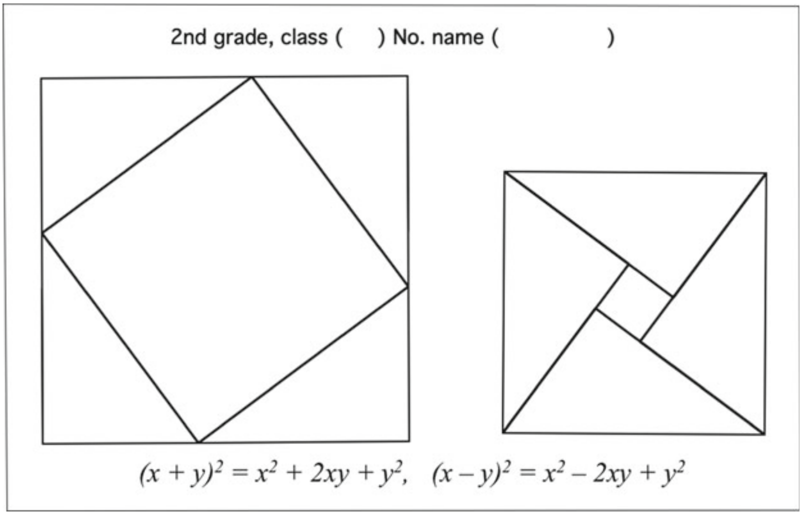
1. ***General information***
   The class is divided in small groups. Students are told that they are expected to do a geometric investigation by means of the computer. Then they receive some instructions how to use the system and how to interact within the groups. The three contexts for approaching the theme are explained in general terms, and the groups are asked to decide for themselves which context they would like to choose.
2. ***Introducing the task***
   When the students start the program three triangles appear on the screen: an obtuse one, a right one and an acute one. The sides of each triangle carry squares: the longest side a square colored red, the smaller sides squares colored green. The students are stimulated to discuss the relationship between the area of the red square and the sum of the areas of the green squares in all three cases. The teacher suggests to draw the squares on graphic paper and to estimate the area. The discussion within the groups and with the whole class should lead to the conjecture of the Pythagorean theorem for right triangles.

3. ***Defining the task***

   The groups are given the following task: Try to find out from the figures and transformations offered by the computer program why the conjectured relationship must hold. Give a written account of your reasoning. Use the prepared worksheets.

   The groups are handed out worksheets that present the essential figures and give some hints for the solution. Groups that have finished their task may switch over to another context.

   Context (1): The group has to express the lengths of the relevant segments and the areas of the relevant figures by means of letters and to derive the Pythagorean theorem by means of algebraic formulae. The worksheet for the context is shown in Fig. 56.

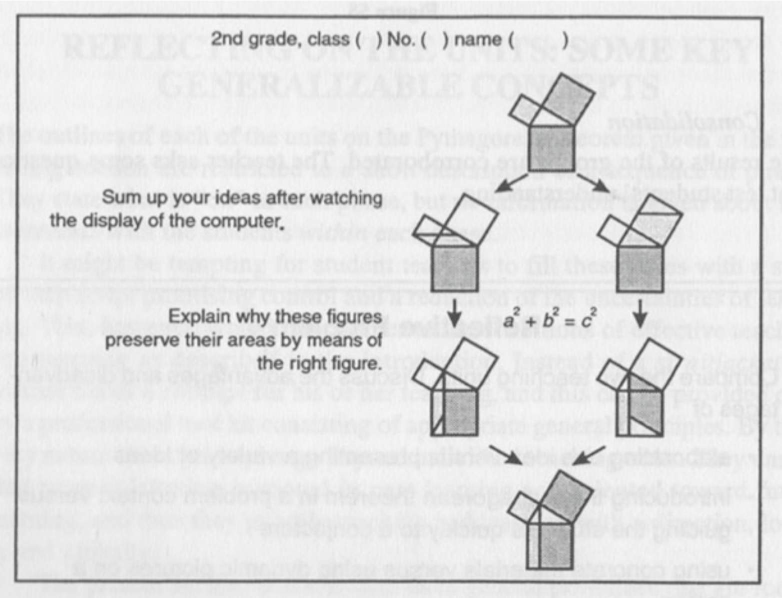


2nd grade, class ( ) No. name ( )

$$(x + y)^2 = x^2 + 2xy + y^2, \quad (x - y)^2 = x^2 - 2xy + y^2$$

Fig. 56

Context (2): The group has to describe and to explain the dynamic version of Euclid's proof (Proof 1*). The worksheet for this context is shown in Fig. 57.



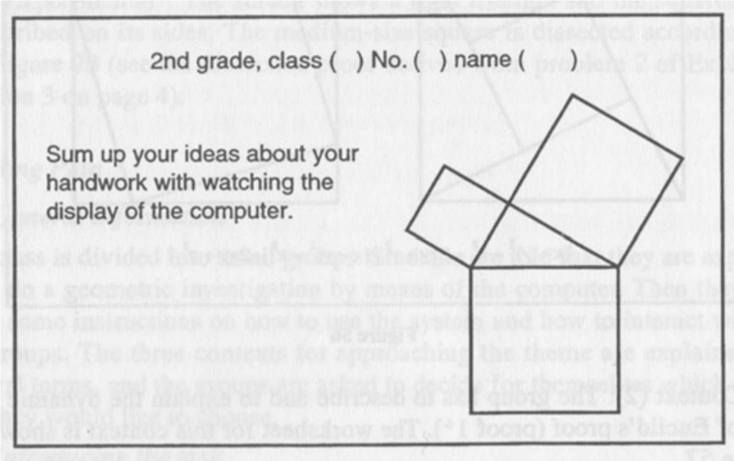
2nd grade, class ( ) No. ( ) name ( )

Sum up your ideas after watching the display of the computer.

Explain why these figures preserve their areas by means of the right figure.

Fig. 57

Context (3): Students are asked to fit the four parts of the medium size square and the small square into the big square and to prove that the five parts fill the big square exactly. The worksheet for the context is shown in Fig. 58.



2nd grade, class ( ) No. ( ) name ( )

Sum up your ideas about your handwork with watching the display of the computer.

Fig. 58

4. ***Consolidation***
   The results of the groups are corroborated. The teacher asks some questions that test students' understanding.

Reflective Problem 4
Compare the two teaching units. Discuss the advantages and disadvantages of

- elaborating one idea versus presenting a variety of ideas
- introducing the Pythagorean theorem in a problem context versus guiding the students quickly to a conjecture
- using concrete materials versus using dynamic pictures on a screen
- formulating a proof orally versus fixing it on a worksheet

Reflective Problem 5
Tom Apostol made a videotape *The Theorem of Pythagoras* that is available from the NCTM. The three main ideas are:

1. Lengths of corresponding sides of similar triangles have the same ratios.
2. Shearing a triangle does not change its area.
3. If the scale for measuring distances is multiplied by a factor k, the area of a figure is multiplied by k^2.

If you have access to this videotape: analyse it in terms of the proofs discussed in this chapter. How are the conceptual relationships represented that are essential for sound proofs but cannot be replaced by pictures ? In which grade would you use the videotape? How would you use it? As an introduction, as an illustration during learning or as a summary?

5 Reflecting on the Units: Some Key Generalizable Concepts

The outlines of each of the units on the Pythagorean theorem given in the preceding section are restricted to a short description of a sequence of phases. They state what is "on" in each phase, but no information is given about how to interact with the students *within* each phase.

It might be tempting for student teachers to fill these holes with a step-by-step script promising control and a reduction of the uncertainties of teaching. This, however, would run counter to the conditions of effective teaching and learning as described in the introduction. Instead of a *straitjacket* the teacher needs a *concept* for his or her teaching, and this can be provided only by a professional tool-kit consisting of appropriate general principles. By their very nature these principles go beyond individual teaching units. They secure that present learning is rooted in past learning and oriented towards future learning, and thus they provide teaching and learning with a direction, locally and globally.

The present section will examine three general principles that are rooted in the analyses and explorations before: the notion of informal proof, heuristic strategies, and the "operative principle."

5.1 Informal Proofs

> A proof becomes a proof only after the social act of "accepting it as a proof." This is as true for mathematics as it is for physics, linguistics or biology. The evolution of commonly accepted criteria for an argument's being a proof is an almost untouched theme in the history of science.
>
> Yuri. I. Manin

A proof of a theorem is a pattern of conceptual relationships linking the statement to the premises in a logically stringent way. In an earlier section we have met a number of proofs of the Pythagorean theorem that vary in the conceptual relationships employed and—even more important for mathematics education—also in their representations. Some of them consist mainly of a text and use a figure just for supporting the text. Others rely heavily upon figures and transformations and contain only a few explaining lines. The proof aimed at in the first teaching unit even uses pieces of cardboard, real displacements and rearrangements of these pieces, and a comment that may be given only orally.

It is of paramount importance for appreciating new developments in the teaching of proofs to understand that the evaluation of different types of proof has been controversial in mathematics *and* in mathematical education over history, particularly in the twentieth century.

For almost two thousand years Euclid's "Elements of Mathematics" dominated mathematics and the teaching of it, and the notion of mathematical proof established in this book was the celebrated peak of mathematical activity. In mathematics education, too, it was admired, emulated as far as possible, and hardly ever questioned, apart from a few outsiders (see, for example, Clairaut 1743).

At the end of the nineteenth century the situation changed fundamentally. Mathematicians and a growing minority of mathematics teachers became dissatisfied with the Euclidean standard for quite different reasons and initiated opposing developments. Mathematicians working in the foundations of mathematics discovered that Euclid unexpectedly had used intuitive assumptions in his logical chains of arguments—for example the assumption that any line intersecting a side of a triangle also intersects at least another side—and they set out to establish a purported level of "absolute" rigor that was to reduce reasoning to a manipulation of symbols and statements according to formal rules. No room was left for intuition. Hilbert's famous book, *Foundations of Geometry*, became the model for the new standard that is perfectly described, for example, in MacLane (1981, 465):

> This use of deductive and axiomatic methods focuses attention on an extraordinary accomplishment of fundamental interest: the formulation of an exact notion of *absolute rigor*. Such a notion rests on an explicit formulation of the rules of logic and their consequential and meticulous use in deriving from the axioms at issue all subsequent properties, as strictly formulated in theorems. ... Once the axioms and the rules are fully formulated, everything else is built up from them, without recourse to the outside world, or to intuition, or to experiment ... An absolutely rigorous proof is rarely given explicitly. Most verbal or written mathematical proofs are simply sketches which give enough detail to indicate how a full rigorous proof might be constructed. Such sketches thus serve to convey conviction—either the conviction that the result is correct or the conviction that a rigorous proof could be constructed. Because of the conviction that comes from sketchy proofs, many mathematicians think that mathematics does not need the notion of absolute rigor and that real understanding is not achieved by rigor. Nevertheless, I claim that the notion of absolute rigor is present.

In mathematics education, on the contrary, a growing number of teachers, supported by a few eminent mathematicians like F. Klein and H. Poincaré, recognized the educational inadequacy of formal systems in general and looked for more natural ("genetic") ways of teaching. Although this movement brought about very nice pieces of "informal" geometry its influence remained quite limited as it failed to develop a global approach to the teaching of geometry comparable in consistency and systematics with the usual programs derived from Euclid. The main difficulty was to conceive a notion of an informal and at the same time sound proof, convincing the mass of teachers.

While up to the 1950s extreme forms of mathematical formalism were mitigated by the pedagogic sensitivity of many teachers who used informal proofs in their teaching, and if only as a didactic concession to their students, the movement of New Maths, influential around the world from the late fifties to the early seventies, sought to introduce mathematical standards of rigor into the classroom without any reduction (see, for example, the excellent analysis in Hanna 1983). This program eventually failed not only because it proved as impracticable, but also, and even more, because mathematical formalism and the idea of "absolute" rigor turned out as mere fictions. Mathematicians became more and more aware that a proof is part of the social interaction of mathematicians, that is of human beings, and therefore not only the discovery but also the check of proofs greatly depend on shared intuitions developed by working in a special field (Davis and Hersh 1983, Chap. 7). The validity of a proof does not depend on a formal presentation within a more-or-less axiomatic-deductive setting, and not on the written form but on the logical coherence of conceptual relationships that are not only to *convince that* the theorem is true, but are to *explain why* it is true. Informal representations of the objects in question are a legitimate means of communication and can greatly contribute to making the proof meaningful.

In a letter submitted to the working group on proof at the 7th International Congress on Mathematical Education, Québec 1992, Yuri Manin, a leading Russian mathematician, described the new view on proof very neatly:

> Many working mathematicians feel that their occupation is discovery rather than invention. My mental eye sees something like a landscape; let me call it a "mathscape." I can place myself at various vantage points and change the scale of my vision; when I start looking into a new domain, I first try a bird's eye view, then strive to see more details with better clarity.

> I try to adjust my perception to guess at a grand design in the chaos of small details; and afterwards plunge again into lovely tiny chaotic bits and pieces.
>
> Any written text is a description of a part of the mathscape, blurred by the combined imperfections of vision and expression. Every period has its own social conventions, and the aesthetics of the mathematical text belong to this domain. The building blocks of a modern paper (ever since Euclid) are basically axioms, definitions, theorems and proofs, plus whatever informal explanations the author can think of.
>
> Axioms, definitions and theorems are spots in a mathscape, local attractions and crossroads. Proofs are the roads themselves, the paths and highways. Every itinerary has its own sightseeing qualities, which may be more important than the fact that it leads from A to B.
>
> With this metaphor, the perception of the basic goal of a proof, which is purportedly that of establishing "truth," is shifted. A proof becomes just one of many ways to increase the awareness of a mathscape...
>
> Any chain of argument is a one-dimensional path in a mathscape of infinite dimensions. Sometimes it leads to the discovery of its end-point, but as often as not we have already perceived this end-point, with all the surrounding terrain, and just did not know how to get there.
>
> We are lucky if our route leads us through a fertile land, and if we can lure other travellers to follow us.

The consequences of this new view for mathematics education can hardly be overestimated (Wittmann and Müller 1990, pp. 36–39). While in the past unjustified emphasis was put on the formal setting of proofs mathematics education is now in a position to exploit the rich repertoire of informal representations without distorting the nature of proof.

In this new framework the use of puzzles in proving the Pythagorean theorem as suggested in the first teaching unit is quite natural. However, it is essential for the soundness of the proof that the decomposition of figures into parts and their rearrangement is accompanied by explanations of *why* the figures fit together in different ways and *what* this means for area. It is the task of the teacher to ensure that the necessary questions are asked and answered by the students. For this interaction with the students the teacher needs a clear understanding of what an informal proof is about.

The use of informal proofs is by no means restricted to geometry. In order to enlighten the difference between formal and informal proofs a bit more we consider the famous theorem on the infinity of primes.

Formal Proof:
Let us assume that the set of prime numbers is finite: $p_1, p_2, \dots, p_r$.
The natural number

$$n = p_1 \cdot p_2 \cdot \dots \cdot p_r + 1$$

has a divisor p that is a prime number, that is, n is divisible by one of the numbers $p_1, \dots, p_r$. From $p|n$ and $p|p_1 \cdot \dots \cdot p_r$ we conclude that p also divides the difference $n - p_1 \cdot \ldots \cdot p_r = 1$. However, $p|1$ is a contradiction to the fact that 1 is not divisible by a prime number. Therefore our assumption was wrong.

Informal Proof:
We start from the representation of natural numbers on the numberline and apply the sieve of Eratosthenes (Fig. 59). The number 2 as the first prime number is encircled, and all multiples of 2 are cancelled as they certainly are not prime numbers. The smallest number neither encircled nor cancelled is 3. The number 3 must be a prime as it is no multiple of a smaller prime. Therefore 3 is encircled and again all multiples of 3 are cancelled. For the same reason as before the first number neither encircled nor cancelled, namely 5, is a prime number. Thus 5 is encircled and all multiples of 5 are cancelled. This procedure is iterated and yields a series 2, 3, 5, 7, 11, ... of prime numbers.

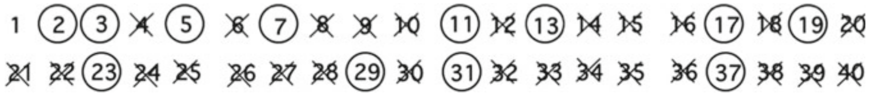


Fig. 59

The infinity of prime numbers will be demonstrated if we can explain why the iterative procedure does not stop. Assume that we have arrived at a prime number p. Then p is encircled and all multiples of p are cancelled. The product $n = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot \ldots \cdot p$ of all prime numbers found so far is a common multiple of all of them. So it was cancelled *at every step*. As no cancellation can hit adjacent numbers the number $n + 1$ has not been cancelled so far. Therefore numbers must be left and the smallest of them is a new prime number.

Comparison of the Two Proofs
First it has to be stated that both proofs are based both are based on similar conceptual relationships. In particular a product of prime numbers increased by 1 plays the crucial role in both proofs. Contrary to the formal proof that works with symbolic descriptions of numbers the informal proof is based on a *visual* representation of numbers on the numberline and on operations on it. In this way the *formal* apparatus can be reduced as some of the necessary conceptual relationships are inbuilt into this representation.

Consequences for Mathematics Teaching
In the past concrete and visual representations of mathematical objects were almost exclusively used for the formation of concepts and for illustrating relationships. Our analyses have shown, however, that appropriate representations are powerful enough to carry sound proofs. This fact opens up to mathematics education a new approach to the teaching of proofs: instead of postponing the activity of proving to higher grades where the students are expected to be mature for some level of formal argument, informal proofs with concrete representations of numbers and geometric figures

can be developed from grade 1. Students can gradually learn to express conceptual relationships more and more formally.

This view on proofs is closely related to Jean Piaget's psychology in which several stages from concrete to formal ones are delineated. Although Piaget's theories have been criticized in many respects the basics of his genetic epistemology are still valid. His emphasis on "operations" as the motor of thinking is of extreme importance for teaching and learning. In Sect. 3 we will investigate the "operative principle".

Although Eratosthenes lived after Euclid it could well be that the sieve was already known to Euclid. As we have seen above that sieve naturally leads to the formal description in the term

$$n = 2 \cdot 3 \cdot 5 \cdot \ldots \cdot p + 1.$$

Exploration 19

The sequence in Fig. 60 indicates a transformation of the squares described on the smaller sides of a right triangle into the square described on the hypotenuse is sometimes offered as a "proof without words". The reader is only invited to look at the figure ("Behold!"). Of course, without any explanations the transformation is nothing but an experimental verification. Elaborate an informal proof by describing the transformations, explaining why they are possible and why area does not change. Hint: See proof 1* for comparison.

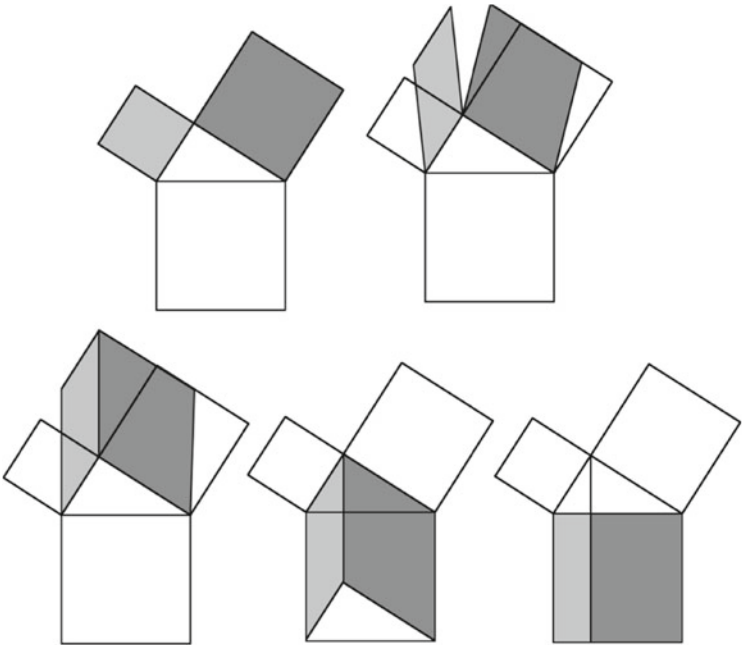


Fig. 60

Exploration 20

Give two proofs of the formula $1 + 3 + \ldots + 2n - 1 = n^2$: an *informal* one based on Fig. 61 and a *formal* one based on mathematical induction.

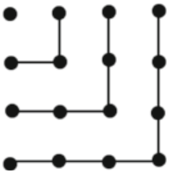


Fig. 61

5.2 *"Specializing"—A Fundamental Heuristic Strategy*

> When we study the methods for solving problems, we notice a different face of mathematics. Actually, mathematics has two aspects; it is the rigorous science of Euclid, but it is also something else. According to Euclid mathematics appears as a systematic, deductive science; but mathematics in the making appears as experimental and inductive. Both aspects are as old as mathematics itself.
>
> G. Polya

The heart of our teaching unit stimulating and controlling all activities is a mathematical problem: *How long is the diagonal of a rectangle?*

The essential step in solving this problem consists of considering a special case—*How long is the diagonal of a square?*—and in generalizing the special solution. It is important to understand this approach not just as a clever trick in the context of the Pythagorean theorem but as a fundamental heuristic strategy widely used in solving mathematical problems.

We owe to G. Polya (1887–1985), the great master of mathematical discovery in this century, a basic revival of *heuristics*, the study of means and methods of problem solving (Polya 1981). Polya's work was taken up and extended by mathematics educators (Mason 1982; Brown and I. and Walter, M. I. 1983; Schoenfeld 1985) and is clearly visible in curriculum developments all over the world (see for example, the items "mathematics as problem solving and as reasoning" in the NCTM-Standards).

Heuristic strategies operate on two levels: They serve to generate new problems out of given ones, and they help to construct solutions of problems out of known results. The two levels, however, are inseparably intertwined: The art of problem posing and the art of problem solving are sides of one and the same medal.

Schoenfeld (1985, 76, 80–81) describes and differentiates the strategy "Specializing" (Strategy S) as follows:

> To better understand an unfamiliar problem, you may wish to exemplify the problem by various special cases. This may suggest the direction of, or perhaps the plausibility of, a solution ... the description of Strategy S given above is merely a summary description of five closely related strategies, each with its own particular characteristics:

Strategy S_1: If there is an integer parameter n in a problem statement, it might be appropriate to calculate the *special cases* when $n = 1, 2, 3, 4$ (and maybe a few more). One may see a pattern that suggests an answer, which can be verified by induction. The calculations themselves may suggest the inductive mechanism.

Strategy S_2: One can gain insight into questions about the roots of complex algebraic expressions by choosing as *special cases* those expressions whose roots are easy to keep track (e.g., easily factored polyomials with integer roots).

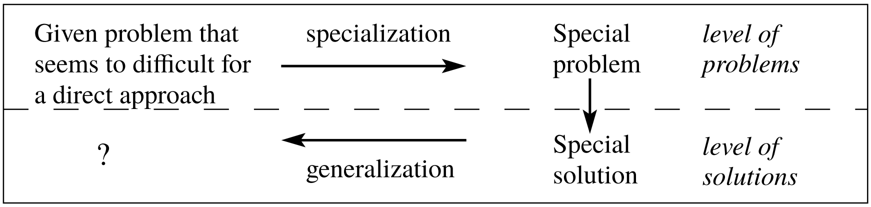
Strategy S_3: In iterated computations or recursions, substituting the particular values of 0 (unless it causes loss of generality) and/or 1 often allows one to see patterns. Such *special cases* allow one to observe regularities that might otherwise be obscured by a morass of symbols.

Strategy S_4: When dealing with geometric figures, we should first examine the *special cases* that have minimal complexity. Consider regular polygons, for example; or isosceles or right or equilateral rather than "general" triangles; or semi- or quarter-circles rather than arbitrary sectors, and so forth.

Strategy S_{5_a}: For geometric arguments, convenient values for computation can often be chosen without loss of generality (e.g., setting the radius of an arbitrary circle to be 1). Such *special cases* make subsequent computations much easier.

Strategy S_{5_b}: Calculating (or when easier, approximating) values over a range of cases may suggest the nature of an extremum, which once thus "determined", may be justified in any of a variety of ways. *Special cases* of symmetric objects are often prime candidates for examination.

The heuristic pattern related to "specializing" can be described as follows:



For further illustration of Strategy S_4, which obviously has been applied in our proof of the Pythagorean theorem, let us consider another example from geometry, Viviani's theorem.This theorem states that the sum of distances of an arbitrary point inside or on the boundary of an equilaterial triangle from the three sides has a constant value independent of the position of P (Fig. 62a–c).

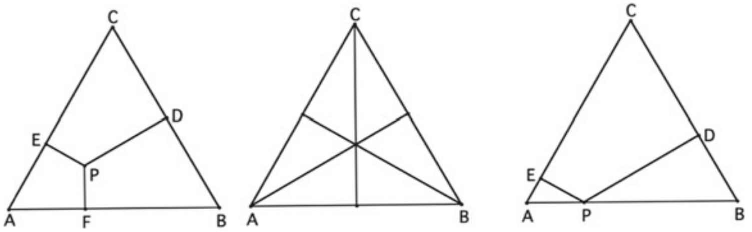


Fig. 62

Students asked to measure the distances and to add them for different points will quickly conjecture this fact. As a direct proof is not near at hand a heuristic approach using Strategy S_4 seems natural.

The complexity is least if P is one of the vertices as then two of the three distances are 0 and the third distance is an altitude of the triangle. In an equilateral triangle all altitudes have equal length h (case 1, Fig. 62b).

The next level of complexity (case 2) is provided by points P on one of the three sides as in this case one distance is 0 (Fig. 62c).

Our goal is to show that the two other distances add up to h.

If P is the midpoint of the side the two distances are equal by way of symmetry (Fig. 63).

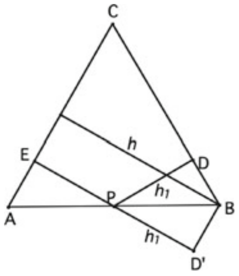


Fig. 63

The reflection at line AB maps PD onto PD'. The 30°-angles around P ensure that D' is on line PE. Because of the right angles at E and D resp. D' lines BD' and AC are parallel and $D'E = 2h_1$ is the distance between them. But this distance is also h. Therefore $2h_1 = h$.

This line of arguments holds also if P is an arbitrary point on a side (Fig. 64).

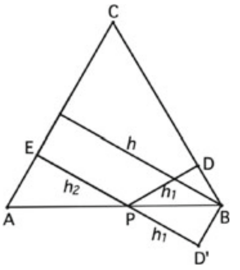


Fig. 64

The general case (P inside of ABC) can be reduced to case 2 (Fig. 65): $A'B'$ is the line through P parallel to AB.

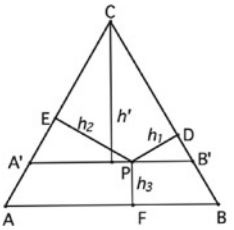


Fig. 65

As the angles at A' and B' are equal to the angles at A and B, triangle $A'B'C$ is equilateral and, according to case 2, $h_1 + h_2 = h' =$ altitude of $A'B'C$. Obviously h' and h_3 add up to h, the altitude of ABC. Therefore $h_1 + h_2 + h_3 = h$. (For an alternative approach following the "What if not ...?" strategy, see Jones/Shaw 1988.)

Our example shows that "specialization" at the level of problems and "generalization" at the level of solutions can be performed in steps: the solution of the problem for an extremely special case is step-by-step transferred to less special cases up to the general case.

The interaction between "specialization" and "generalization" is often used to generate new knowledge in the following way: One tries to generalize a problem that has been solved (possibly in different ways). If a reasonable generalization has been found one attempts to generalize the solution(s).

Pattern of generalization:

?	← generalization	Problem
?	← generalization	Solution(s)

This heuristic strategy is particularly fruitful for the Pythagorean theorem and leads to the discovery of two important generalizations:

Law of cosines:
The sides a, b, c and the angles a, b, c of an arbitrary triangle are related by the following formulae:

$$a^2 + b^2 = c^2 \ + \ 2ab\cos\gamma$$
$$b^2 + c^2 = a^2 \ + \ 2bc\cos\alpha$$
$$c^2 + a^2 = b^2 \ + \ 2ca\cos\beta$$

General Pythagorean theorem:
If similar figures are described on the sides of a right triangle then the sum of the areas of the two smaller figures is equal to the area of the third figure.

For an excellent heuristic analysis of these generalizations the reader is referred to *The Art of Problem Posing* by Stephen Brown and Marion Walter (1983, 44–61, 112–116).

Exploration 21

The midpoints of the sides of an arbitrary quadrilateral are the vertices of a new quadrilateral. What do you conjecture about the shape of this midpoint-quadrilateral? How long are its sides?
Prove your conjecture by applying the strategy "specialization".

Exploration 22

Generalize Problem 2 of Exploration 3.

5.3 *The Operative Principle*

> It would be a great mistake, particularly in mathematics education, to neglect the role of operations and always to remain on the level of language. ... The initial role of operations and logico-mathematical experience, far from hindering the later development of deductive thought, constitutes a necessary preparation.
>
> J. Piaget

In discovering the Pythagorean theorem and in establishing a proof as envisaged in the teaching unit students have to "play around" with figures: Squares and rectangles are dissected, the pieces are arranged in various ways, a hole is filled, and so forth.

The teacher of mathematics must be aware that this activity offers by no means just an ad hoc approach to the Pythagorean theorem but that it reflects the natural functioning of our cognitive system. According to the constructivist view of learning, knowledge is neither received from environmental sources (that is from structures considered as inherent in reality or structures offered by the teacher) nor unfolds simply from inside. Knowledge is *constructed* by the individual through interacting with the environment: the individual operates upon the environment and tries both to assimilate the environment to his or her mental structures and to accommodate the latter to the external requirements.

Let us illustrate this goal-directed "playing around" by means of some examples.

Episode 1: During a christmas party a 1.5-year-old is sitting on the legs of his father at a table with candlelights. He gazes at a candle burning in front of him, but out of his reach. Suddenly a child on the other side of the table bends over the table and blows the candle out. The boy observes the event carefully and notes how somebody else lights the candle again. Now it is he who wants to blow the candle out: he hisses—the candle is still burning, he reinforces his hissing sound, again without success, he growls, he moves his body, first towards the candle, then aside, he hits the table with his hands and moves them around and so forth. All cognitive schemas available to him are tested, however, without success. After 15 minutes the boy loses his interest.

Episode 2: Two twelve-year-olds play the following game of strategy (Fig. 66).

Fig. 66

One of the players has red counters, the other blue ones. They take turns to fill the row from 1 to 10 successively with counters. Each player may add one or two counters of his colour. The player first arriving at 10 is the winner.

First the students play more or less randomly. Then they discover that 7 is a favourable position: the player arriving at 7 can also arrive at 10: If the opponent adds 1 counter, then 2 counters lead to 10. If the opponent adds 2 counters, then 1 counter is sufficient to cover 10. By trying out different moves and by evaluating them the students discover that 4 and 1 are also favourable positions, and that the player starting the game has a winning strategy.

Episode 3: A student teacher tries to solve the following geometric problem by means of The Geometer's Sketchpad or Geogebra: Given lines g, h and circle k construct a square $ABCD$ such that A lies on g, B and D on h, and C on k (Fig. 67). First she draws g, h and k. Then she chooses A on g as a moving point. She recognizes that the choice of A determines B and D on h as the foot F of the perpendicular l dropped from A to h must be the midpoint of the square. The student constructs points B and C as images of A under rotations with center F and angles 90° and −90°. Next she recognizes that A is mapped to C by means of a rotation with center F and angle 180°. But C does not lie on circle k. In order to fulfill this requirement the student moves A along g, back and forth. B, C and D move correspondingly and it is easy for her to maneuver C on k.

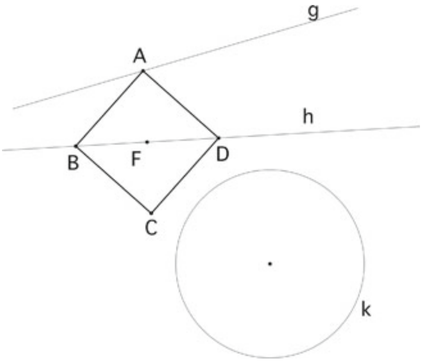


Fig. 67

Actually, there are two solutions in this case. When performing the movement a second time the student suddenly observes that A and C move symmetrically with respect to h (Fig. 68). This leads her to the following solution of the problem: Line g is reflected on h into g'.The intersections of g' and circle k are possible positions for vertex C. Dropping the perpendicular from C to h and intersecting it with g gives the corresponding vertex A. B and D can be constructed as above.

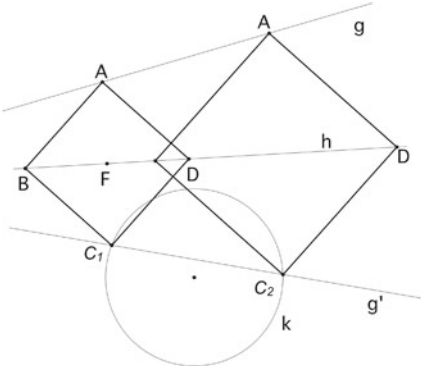


Fig. 68

Each of the three episodes illustrates an important aspect of Piaget's view: The searching individual acts upon objects and observes the effects of his or her actions (episode 1). Known effects are used for anticipating paths to certain goals (episode 3). Knowledge is not a ready-made matter, but it is constructed by the individual through interaction with reality (episode 2).

This "operative" approach ranges from everyday situations to more and more abstract and complex mathematical situations, from concrete objects to symbolically represented objects, and thus it is essential for the whole mathematical curriculum.

For illustration, again a few examples.

Example 1 (*Primary level: Addition and Subtraction*) Problem: The sum of two numbers is 32, the difference is 8. Which are the numbers?
To solve this problem the numbers are represented by counters of different colours (Fig. 69).

Fig. 69

Here 16 red and 16 blue counters make 32, but the difference is 0. Replacing a blue counter by a red one leads to $17 + 15 = 32$, $17 - 15 = 2$. Repeating this operation two more times gives

$$19 + 13 = 32, \;\; 19 - 13 = 6, \;\; 20 + 12 = 32, \;\; 20 - 12 = 8$$

Example 2 (*Secondary level: Symmetric figures*)

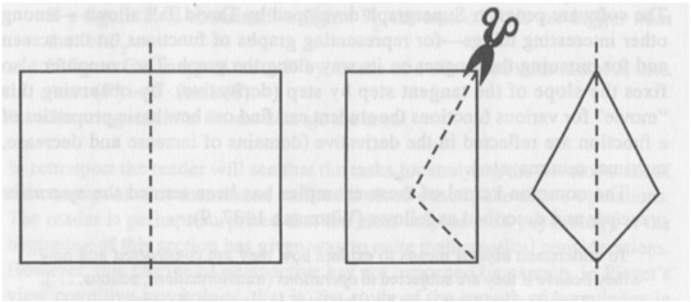


Fig. 70

A rectangular piece of paper (Fig. 70) is folded along a line of symmetry and cut along the dotted lines. The shaded triangle is unfolded and leads to a special quadrilateral, a kite.

Questions:
- Which properties are imprinted into the kite by this generating process?
- Which forms can a kite have?
- How to cut in order to make all sides equally long?
- Can a square be generated in this way?

To answer these questions students will have to fold, cut, check, vary the attempts, check again until they arrive at the answers.

Example 3 (*Secondary level: Quadratic functions*) The graph of a quadratic function is typically derived from the standard parabola, the graph of the function $y = x^2$, by means of four basic *geometric* transformations that model *algebraic* transformations of the functions:

Algebraic transformation			*Geometric transformation*
$y = x^2$	into	$y = ax^2$	Affine dilatation of the standard parabola with factor a along the y-axis
$y = ax^2$	into	$y = -ax^2$	Reflection at x-axis
$y = ax^2$	into	$y = a(x - c)^2$	Translation by c along the x-axis
$y = a(x - c)^2$	into	$y = a(x - c)^2 + d$	Translation by d along the y-axis

Example 4 (*(College): Derivative*) The software program Supergraph developed by David Tall allows—among other interesting things—for representing graphs of functions on the screen and for pursuing the tangent on its way along the graph. The computer also fixes the slope of the tangent step by step (derivative). By observing this “movie” for various functions the student can find out how basic properties of a function are reflected in the derivative (domains of increase and decrease, maxima, minima, etc.).

The common kernel of these four examples has been termed the *operative principle* and described as follows (Wittmann 1987, 9):

> To understand *objects* means to explore how they are *constructed* and how they *behave* if they are subjected to operations (transformations, actions, ...).
>
> Therefore students must be stimulated in a systematic way
>
> (1) to explore which *operations* can be performed and how they are related with one another,
>
> (2) to find out which *properties* and relationships are imprinted into the objects through construction,
>
> (3) to observe which effects properties and relationships are brought about by the operations according to the guiding question “What happens with ..., if ...?”

In this formulation the nature of the “objects” has deliberately been left open. Therefore the operative principle has a wide range of applications.

It is not by chance that examples 3 and 4 employ the computer. In fact the computer, if properly used, is the ideal device for making the operative principle practical.

Through the lens of the operative principle the concept of area appears in the following operative setting:

The “objects” in question are geometric figures. These figures can be changed by a great variety of “operations,” for example, reflections, translations, rotations, dilatations, shearing motions, decompositions, extensions, reductions... The standard questions are: *What happens* with the area of a figure *if* the figure is reflected, translated, decomposed, extended ...?
Answers:

Area behaves *invariant* under rigid motions,
additive under decompositions,
monotone under extensions,
quadratic under dilatations and
invariant under shearing motions.

In other words:

Congruent figures have the *same* area.
The area of a composite figure is the *sum* of the areas of its parts.
If a figure F_1 is contained in figure F_2, the area of F_1 is *not bigger than* that of F_2.
If figure F is mapped on to F’ by means of a dilatation with factor k, then Area (F’) $= k^2 \cdot$ Area (F).
Shearing motions do *not change* the area.

In retrospect the reader will see that the tasks for studying the development of the concept of area involved exactly the above operations. The reader is perhaps surprised that the clear emphasis on psychology at the beginning of this section has given way to quite mathematical considerations. However, this change of perspective has not happened by chance. In Piaget’s view cognitive *psychology*, that is, the study of the growth of knowledge in individuals, is strongly related to *epistemology*, that is, the study of growth and structure of scientific knowledge.

The “operative” view at cognition, learning and teaching is also strongly related to the notion of proof. In Sect. 1 it was stated that a proof is a logical chain of conceptual relationships. Now we can put it a little more precisely: In a proof objects are presented and introduced that are *constructed* in characteristic ways, and these objects are subjected to certain *operations* such that known effects arise. It is from these constructions and operations that the essential conceptual relationships flow on which the proof is based.

For illustration let us consider Proof 2 of the Pythagorean theorem. The proof starts with constructing an appropriate figure (Fig. 71). Then certain parts of the figure are analyzed whereby at some places operations appear.

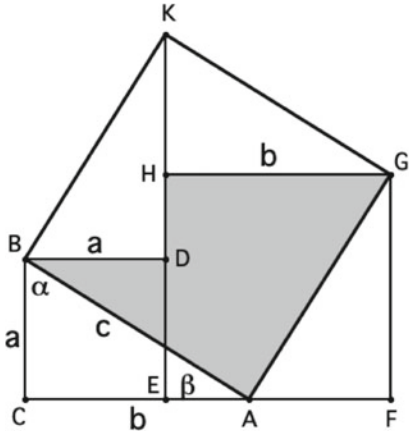


Fig. 71

Objects	*Relationships imposed on the objects by construction or by operation*
Triangle ABC:	$\alpha + \beta = 90°$
Segment AF:	$AF = a$
Segment DK:	$DK = b$
Triangles ABC, GAF, GHK, KBD:	all congruent (triangle ABC can be laid upon the others)
Quadrilateral $AGKB$:	square
Hexagon $BCFGHD$	consisting of squares $BCED$ and $EFGH$ and square $AGKB$
	The square and the hexagon are equidecomposable, and therefore of equal area (three parts covering the hexagon can be rearranged to cover the square)

Exploration 23

What are “objects,” “operations” and “effects” in

1. the Bhaskaran puzzle proof of the Pythagorean theorem (see p. 133, Fig. 50),
2. Clairaut’s approach (see Figures 15 to 19 and Exploration 11),
3. the three episodes and the four examples of the present section?

6 Appendix: Solutions to the Problems in Exploration 3

Problem 1 This is a typical example for using the Pythagorean theorem. First, the diagonal d of the rectangular base $ABCD$ is calculated, $d^2 = a^2 + b^2$. Then the Pythagorean theorem is applied once more: the triangle ACP with sides c, d and s is also right. Therefore

$$s^2 = d^2 + c^2 = a^2 + b^2 + c^2, \text{ or } s = \sqrt{a^2 + b^2 + c^2}.$$

Problem 2 The problem seems to call for the Pythagorean theorem, and in fact it is possible to solve it by calculating the side of the shaded square in several steps by means of the Pythagorean theorem and similarity arguments. However, the Pythagorean theorem is not necessary. The figure can be embedded into a square lattice (why?). By comparing the parts of the resulting dissection (see Fig. 72) one sees that the original square is five times the area of the shaded square.

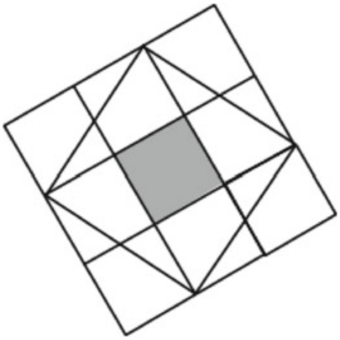


Fig. 72

The new figure, however, is close to figures related to the Pythagorean theorem, for example to the figures in Proof 3 and in Exploration 8. If we combine the parts of the original square in an appropriate way (see Fig. 73), we touch the idea of a new dissection proof of the Pythagorean theorem: by starting from the midpoints of its sides a big square can be dissected into a small square and four congruent rectangles. The latter can be recombined to make a square whose sides are twice the length of the small square (see Fig. 74).

It is important to see the Fig. 71 not as static but as a dynamic network of elements that are connected by relationships.

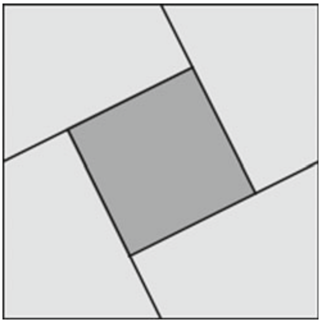


Fig. 73

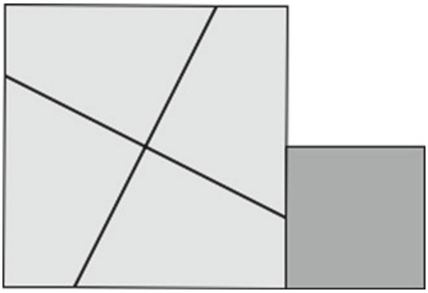


Fig. 74

The transition from Figs. 72, 72 and 74 can also be made with arbitrary squares. All one has to do is to decompose the sides of the larger square into two segments whose difference is the side of the smaller square. Check it and you have the idea of a new proof of the Pythagorean theorem! Explain Fig. 75, which is used by the Japanese teaching unit mentioned earlier (worksheet (3)).

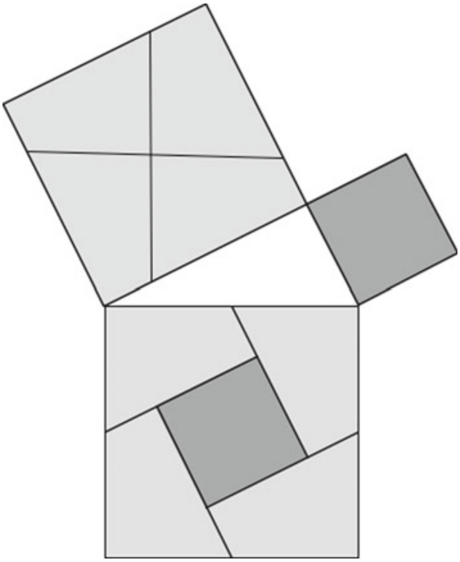


Fig. 75

Problem 3 Assume that the car is 4.60 m long and that the distances to the adjacent cars are 0.30 m each. Then the “length” available for the car is 4.60 m $+ 2 \times 0.3$ m $=$ 5.20 m. In order to move the car out of the lot without too much trouble the diagonal d of the car should be a little bit smaller than the available length 5.20 m (see Fig. 76).

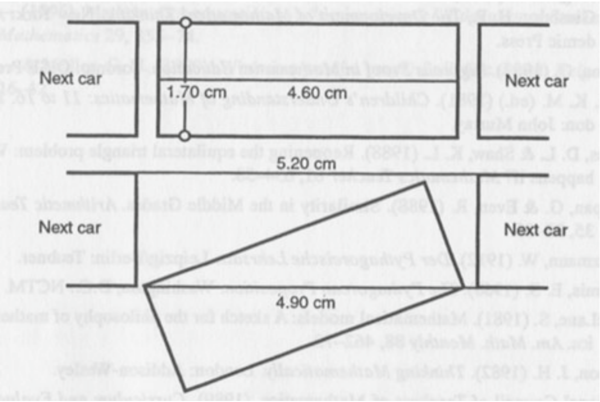


Fig. 76

Application of the Pythagorean theorem leads to

$$d = \sqrt{(4.60)^2 + (1.70)^2}\,\mathrm{m} \approx 4.90\mathrm{m}.$$

4.90 m is 30cm smaller than the available length. So it is possible to move the car out.

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Chapter 8
Standard Number Representations in the Teaching of Arithmetic

Abstract The paper describes the specific approach towards the choice and use of number representations as developed by the project Mathe 2000.

The project Mathe 2000 founded in 1987 at the University of Dortmund is a developmental research project that is based on a conception of mathematics education as a design science. In the past the project has been concerned with developing theoretical concepts and practical materials for the teaching of mathematics at the primary level, including an innovative textbook series. However, the project represents a comprehensive view of mathematics teaching and will be extended to the secondary level. As a special feature of the project, design, empirical research, pre-service and in-service teacher training, and public relations are closely linked and pursued simultaneously. Essential for this approach is the establishment of a "theory-practice network" bringing together all partners of the educational system. Here the "Handbook for Practicing Skills in Arithmetic" (2 vols.) has been playing a fundamental role as a basic reference.[1]

The present paper is intended to describe and illustrate a specific feature of this project, namely the development of a "grammar of non-symbolic representations" as suggested in Wittmann 1988. The area that is chosen here is arithmetic, one of the basic areas of mathematics teaching.

The first section of this paper presents ten principles which describe basic views on teaching and learning.

In the second section, the epistemological nature of number representations is elaborated in contrast to their use as methodical instruments in traditional didactics. It will be shown that "representation" is a fundamental idea of mathematics.

The third section is devoted to the problem of selecting standard number representations for teaching arithmetic to which symbolic notations are usually related.

Finally, the use of standard representations is illustrated by means of some teaching units that also represent fundamental ideas of arithmetic.

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[1] For a broader discussion of the Mathe 2000 approach to primary mathematics see Becker and Selter 1996.

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1 Principles of Learning and Teaching

In spite of differences in detail, most mathematics educators around the world might share the following basic views of learning and teaching: Mathematical concepts are neither innate nor readily acquired through experience and teaching. Instead the learners have to reconstruct them in a continued social process where primitive and only partly effective cognitive structures which are chequered with misconceptions and errors gradually develop into more differentiated, articulated and coordinated structures which are better and better adapted to solving problems. Teachers cannot expect concepts to be readily transferable from teacher to student. As Johannes Kühnel put it neatly at the beginning of this century, the main role of the student is characterized by "activity", not by "receptivity", and correspondingly the main role of the teacher by "organisation", not by "instruction". In interacting with students, the teacher must have a feeling for students' ways of thinking and help them to reconstruct their conceptual structures on a higher level.

The diagram in Fig. 1 is an attempt to capture this "genetic" or "developmental" view on teaching and learning in a system of ten principles in order to make it practical for the design of teaching. The author is well aware that systems of this kind are always scholastic to some extent. Nevertheless, he has found the diagram useful for keeping balance in coping with the many aspects of learning and teaching.

Fig. 1 Principles of learning and teaching

Explanation of the diagram: The vertices of the triangle are linked to the old didactic triad teacher, subject matter, learner and mark the epistemological, the psychological and the social corners of the diagram: In order to organize learning situations that stimulate students' activities and social interactions (upper vertex) the teacher has to mediate between the genesis of mathematical knowledge on the one hand (left vertex) and the student's developing cognitive repertoire on the other hand (right vertex). The operative principle, derived from Jean Piaget's epistemology and psychology, integrates epistemological, psychological and social aspects, and thus occupies the central position by right.

The "spiral principle" (Bruner 1960, Chap. 3), Vygotsky's principle of the "zone of proximal development" (Wertsch 1985) as well as the principle of "natural differentiation"[2] refer to different levels in the development of knowledge of which the teacher has to be aware.

Three other principles are related to the problem of representing knowledge: one of them speaks in favor of a careful selection of basic representations. Another one recommends a "progressive schematisation" in the representation of knowledge during the learning process (Treffers 1987). The third states that it is impossible for the student to understand even concrete and visual representations directly and postulates their interactive exploration (Schipper 1982; Voigt 1989; Lorenz 1992; Cobb et al. 1992).

Finally, the diagram involves another grouping of the ten principles: the three principles at the left corner together with the central principle form the four *epistemological* principles. Similarly, we have four *psychological* principles at the right comer and four *social* principles on the upper corner. The operative principle belongs to all three groups because of its integrative character.

For the present paper, it is important to note that the problem of representing knowledge plays a fundamental role in the concept of learning and teaching as expressed in the diagram.

2 The Epistemological Nature of Number Representations

In order to clarify the background of the principles of Fig. 1 concerned with "representations", it is crucial to examine the history of teaching aids as well as the role of representations in mathematics itself in some detail.

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[2] "Natural differentiation" means that the individual student, not the teacher, decides which of the offered tasks he or she should choose and elaborate on.

2.1 Notes on the History of Number Representations: From Tools of Teaching to Tools of Learning

The use of teaching aids was postulated for the first time in full weight by Comenius, one of the arch-fathers of didactics, in the 17th century. In his classic "Didactica Magna", published in 1657, he formulated the "golden rule" for all teaching (Comenius 1923, 184–187):

> From this a golden rule for teachers may be derived. Everything should, as far as is possible, be placed before the senses. For this there are three cogent reasons. Firstly, the commencement of knowledge must always come from the senses (for the understanding possesses nothing that it has not first derived from the senses).
>
> Secondly, the truth and certainty of science depend more on the witness of the senses than on anything else. For things impress themselves directly on the senses, but on the understanding only mediately and through the senses. It follows, therefore, that if we wish to implant a true and certain knowledge of things in our pupils, we must take especial care that everything be learned by means of actual observation and sensuous perception.
>
> Thirdly, since the senses are the most trusty servants of the memory, this method of sensuous perception, if universally applied, will lead to the permanent retention of knowledge that has once been acquired. If the objects themselves cannot be procured, representations of them may be used. Copies or models may be constructed for teaching purposes. For every branch of knowledge similar constructions (that is to say, images of things which cannot be procured in the original) should be made, and should be kept in the schools ready for use. It is true that expense and labor will be necessary to produce these models, but the result will amply reward the effort.

What Comenius had in mind was to replace the purely verbal teaching practiced in the Middle Ages. His efforts were mainly directed towards elementary science. A famous example for his intentions is his classic textbook "Orbis pictus". The teaching of mathematics is not touched by Comenius, as mathematics, according to the then prevailing Platonic view, belonged to the realm of intellectual and spiritual ideas and therefore was not accessible to concrete or visual representation and perception.

In the 18th century this situation changed completely, when Kant in his philosophic system assigned a quite different status to mathematics. According to Kant, mathematical knowledge is "synthetic", that is depends on basic perceptions of space and time. In the introduction of his "Critique of pure reason", Kant explains this new conception of mathematics in a way which is very close to mathematics teaching (Kant 1943, 9–10):

> We might, indeed, at first suppose that the proposition $7 + 5 = 12$, is a merely analytical proposition, following (according to the principle of contradiction), from the conception of the sum of seven and five. But if we regard it more narrowly, we find that our conception of the sum seven and five contains nothing more than the uniting of both sums into one, whereby it cannot at all be cogitated what this single number is which embraces both. The conception of twelve is by no means obtained by merely cogitating the union of seven and five; and we may analyze our conception of such a possible sum as long as we will, still we shall never discover in it the notion of twelve. We must go beyond these conceptions, and have recourse to an intuition which corresponds to one of the two- our five fingers, for

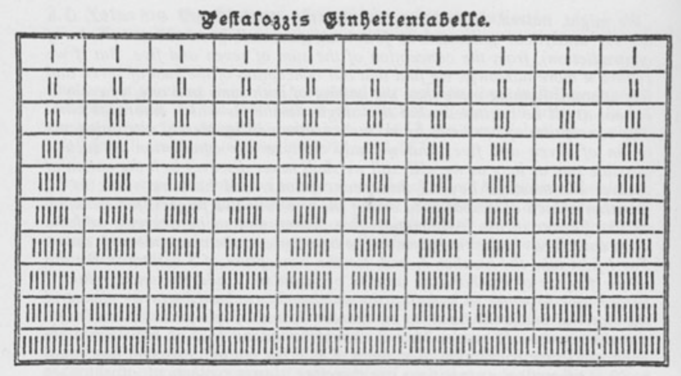
> example, or like Segner in his "Arithmetic", five points, and so by degrees, add the units contained in the five given in the intuition, to the conception of seven. For I first take the number 7, and, for the conception of 5 calling in the aid of the fingers of my hand as objects of intuition, I add the units, which I before took together to make up the number 5, gradually now by means of the material image my hand, to the number 7, and by this process, I at length see the number 12 arise. That 7 should be added to 5, I have certainly cogitated in my conception of a sum = 7 + 5, but not that this sum was equal to 12. Arithmetical propositions are therefore always synthetical, of which we may become more clearly convinced by trying large numbers. For it will thus become quite evident, that, turn and twist our conceptions as we may, it is impossible, without having recourse to intuition, to arrive at the sum total or product by means of the mere analysis of our conceptions.

Kant's statement

> Concepts without intuitive perception are empty, intuitive perceptions without concepts are blind

became the slogan for a big movement among educators who set out to ground teaching on visual representations.

One of the first to apply Kant's view to the teaching of mathematics was Pestalozzi. In his "ABC der Anschauung" he attempted to teach basic facts about natural numbers and fractions through precisely defined chains of exercises related to the "Table of units" (Fig. 2) and to the decomposition of a square (Pestalozzi 1803). Although in theory perception was related to activity ("Selbsttätigkeit") Pestalozzi's practical proposals lacked far behind his postulate, as we understand it nowadays.



Peſtalozzis Einheitentabelle.

I	I	I	I	I	I	I	I	I	I
II	II	II	II	II	II	II	II	II	II
III	III	III	III	III	III	III	III	III	III
IIII	IIII	IIII	IIII	IIII	IIII	IIII	IIII	IIII	IIII
IIIII	IIIII	IIIII	IIIII	IIIII	IIIII	IIIII	IIIII	IIIII	IIIII
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Fig. 2 Pestalozzi's "Table of units"

Along the same lines Tillich wrote a textbook for teaching arithmetic which was based on a kit of rods, a precursor of the Cuisenaire rods (Tillich 1806). Beyond mere perception an element of "activity" on part of the student is implicit in Tillich's approach. It was Froebel who some years later made this element explicit in his theoretical approach to cognition. However, it is important to note that here "activity" was understood as completely prescribed behavior. No room was left to the student's own initiative, as is shown by the following section "Representation and perception of the number series as a continuous whole" from Froebel's course (Froebel 1826/1966, 289–290, transl. E.Ch.W.). Essentially the basic representation is nothing but one column of Pestalozzi's "Table of units":

> Count from One to Ten and each time draw as many vertical strokes (of a certain length) as the number word indicates, that is One |, Two ||, Three |||, etc., one below the other.
> (One) … |, (Two) … ||, (Three) … |||, (Four) … ||||, etc.
> Have you done so? What have you done?
>
> We have counted from one to ten, and each time , etc.
>
> Well! You have represented the natural series of all numbers from one to ten. What have you represented?
> Emphasizing, perceiving and getting aware of the interaction between word and set, number itself:
>
> – starting from the number words:
>
> Teacher and students recite in unison by pointing to the represented series:
> One is | (one One), Two is || (two Ones), Three is ||| (three Ones), etc.
>
> – starting from the number or set:
>
> Teacher and students recite in unison by pointing to the sequence
> | is One, || is Two, ||| is Three, etc.
> Word and set merge, appear as one thing, number is perceived in its pure form:
> | One is One, || Two is Two, ||| Three is Three, etc.
> Recitation by teacher and students as before.

During the 19th and 20th century a wide variety of teaching aids for arithmetic was invented. Although many of these inventions aimed at student "activity" the restrictions from the times of Tillich and Froebel were never overcome. Teaching aids remained tools in the hands of the teacher and were subordinated to didactic systems which were based on the empiricist credo that knowledge is something to be transmitted from teacher to student. It was firmly believed that teaching aids work the better the more their use is prescribed by the teacher in all details. This belief was taken over by theories of learning arising in the 19th century which recognized the insufficiency of the "perception channel" for learning and introduced the "activity channel". These advanced empiricist theories were further elaborated in the 20th century and exerted a great influence on didactics and on the teaching practice'. A good example from our time is Galperin's theory of learning and teaching in which the "reflection of the objective reality in the mind" is pursued by activities prescribed in detail step by step (cf., Gravemeijer 1994, Sect. 2, for an excellent analysis of the traditional use of manipulatives and its limitations).

A typical application of this narrow didactic view is provided by the "transition beyond ten" in traditional German didactics for grade 1. In order to calculate $7 + 5$ Cuisenaire rods (or counters) must be arranged in a definite way (Fig. 3) and the calculation must follow prescribed steps:

$$7 + 3 = 10, \quad 5 - 3 = 2, \quad 10 + 2 = 12.$$

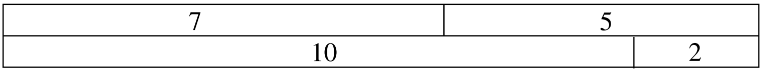

Fig. 3 Using Cuisenaire rods for deriving results

In the past 20 years "constructivist" conceptions of learning and teaching have gained ground and have shed a quite different light on "teaching" aids. In Europe this movement was very much influenced by Piagetian psychology. Already in the late sixties Jean Piaget stated the shortcomings of traditional visual methods very clearly (Piaget 1970, 71–72):

> One of the causes of the slowness with which the active methods have been adopted is the confusion that sometimes occurs between the active methods and the intuitive methods. A certain number of pedagogues in fact—and often in the best possible faith - imagine that the latter are an equivalent of the former, or at least that they produce all the essential benefits that can be derived from the active methods.
>
> We are faced here, moreover, with two distinct confusions. The first, which has already been mentioned, is that which leads people to think that any "activity" on the part of the student or child is a matter of physical actions, something that is true at the elementary levels but is no longer so at later stages when a student may be totally "active", in the sense of making a personal rediscovery of the truths to be acquired, even though this activity is being directed toward interior and abstract reflection.
>
> The second confusion consists in believing that an activity dealing with concrete objects is no more than a figurative process, in other words nothing but a way of producing a sort of precise copy, in perceptions or mental images, of the objects in question. In this way it is forgotten that knowledge is not at all the same thing as making a figurative copy of reality for oneself, but that it invariably consists of operative processes leading to a transformation of reality, either in actions or in thought. It is also forgotten that the experience brought to bear on the objects may take two forms, one of which is logico-mathematical and consists in deriving knowledge, not from the objects themselves, but from the actions as such that modify the objects.
>
> Since all this has been forgotten, the intuitive methods come down, quite simply to a process of providing students with speaking visual representations, either of objects or events themselves, or of the result of possible operations, but without leading to any effective realization of those operations. These methods, which are, moreover, traditional, are continually being reborn from their own ashes and do certainly constitute an advance to purely verbal or formal teaching techniques. But they are totally inadequate in developing the child's operative activity, and it is only as a result of a simple confusion between the figurative and the operative aspect of thought that it has been believed possible to pay tribute to the ideal of the active methods while at the same time giving concrete form to the subject matter of education in this purely figurative guise.

In his critical analysis of the mass of new materials and diagrams introduced into the teaching of mathematics by "New Math", Schipper discovered an important fact (Schipper 1982): Children do not understand these representations immediately nor automatically. On the contrary, they must learn them as a kind of additional subject matter.

In the meantime this fact has been validated by a variety of research findings (Radatz 1986; Voigt 1989; Jahnke 1989; Lorenz 1992). Krauthausen (1994) has summarized the new view on concrete and visual representations, diagrams etc. in six propositions (Krauthausen 1994, 30–35):

1. *What primarily counts are mental images of concepts. Visual representations can support these to some extent.*
2. *Mental images are not just copies of external representations, but they are formed by the constructive activity of the individual.*
3. *These constructions are idiosyncratic, that is, they are determined by the experiences and personal knowledge of the individual.*
4. *Concrete and visual representations are no, speaking pictures', they do not fulfil the expected function as carriers of mental images per se.*
5. *Concrete and visual representations are neither only aids for the so-called'slow learners' nor is their use restricted to the early steps of the learning process. They are important for all children and they are useful for the whole duration of the learning process.*
6. *Concrete and visual representations are not automatically the better, the more specifically they represent the intended concept. ,Perfect' representations can be counterproductive. In order to fulfill their function good representations must involve a certain vagueness.*

In this new view representations of mathematical structures are no longer considered as tools of the teacher for transmitting knowledge, but as tools of the learner for doing mathematics. Their status is no longer a didactic, but an epistemological one (Wittmann 1993).

Representations of knowledge develop with the cognitive repertoire of the individual. They have to be constructed and re-constructed in an extended interactive process as a kind of language and as a field for exploration. It is through applying and testing these representations in new contexts that the individual understands and uses them better and better.

2.2 Representations in Mathematics

The shift from didactics to epistemology draws the attention to the true origin of representations, namely to mathematics itself. Kaput (1987) and Dörfler (1991) have very clearly pointed out the fundamental role which is played by representations of mathematical objects in mathematical research. Even if systematic-deductive presentations of mathematical theories do not point it out explicitly, mathematical theories do not only involve concepts, theorems, and algorithms, but also the construction of objects to which they pertain. These constructions form a "quasi-reality" which allows for experimentally investigating concepts, conjectures and proofs. So within the development of a theory there is a continuous interplay between descriptions and constructions. For example, the exponential function can be described as a homomorphism of the additive group of real numbers into the multiplicative group of real numbers, and it can also be constructed by defining its values step by step for natural numbers, integers, fractions and irrational numbers. In a similar way, a group can be defined by means of axioms, and it can be constructed as a permutation group. Within every theory we also find attempts to characterize classes of objects by showing how they are constructed out of well-known special objects. Theorems of this kind are called representation theorems (Kaput 1987). A good example is provided by the complete classification of finite simple groups achieved in the early eighties.

In higher mathematics representations are in general symbolic. However, the early history of mathematics demonstrates very clearly the basic role of counters ("calculi") in the development of arithmetic (cf., Damerow/Lefèvre 1981). It is interesting to note that the ancient Greeks discovered and proved the first theorems on even, odd and figurate numbers by forming appropriate patterns of counters (cf., Becker 1954, 34–40), and it is equally interesting to realize that eminent research mathematicians emphasize the explanatory power of these patterns for the foundations of arithmetic even today (cf., Penrose 1994, 48–50).

Considered through the epistemological lens, representations have an amphibian-like status: On the one hand they are quasi-real or, as in the case of counters, even real, on the other hand they carry theoretical relationships. As a consequence the use of representations is not restricted at all to founding or illustrating concepts. On the contrary: Representations can be used and are used in the full process of making mathematics including mathematizing, discovering, reasoning, and communicating. The rise of experimental mathematics due to efficient computer software bears witness to this fact.

Because of their amphibian-like status representations can be used to model real situations and they can be used to model mathematical structures. In the first case they are "more abstract" than what they model, in the second case they are "more concrete".

By working with representations of mathematical objects, even proofs of general statements become possible, as explained in Piaget's epistemology: According to Piaget, mathematical knowledge is not derived from the objects themselves, but from operations with objects in the process of reflective abstraction ("abstraction réfléchissante", Beth/Piaget 1961, 217–223). When it is intuitively clear that the operations applied to a special object can be transferred to a class of objects the relationships based on these operations are general. An instructive example is given by Dress (1974) who describes in detail how he found and proved a theorem on the Burnside ring of finite groups by elaborating the special case A_5, the alternate group of order 5.

In the process of learning the interplay between formal descriptions of mathematical structures by means of symbols ("signs") on the one hand and their quasi-real representations ("referents" or "objects") on the other hand is reflected in the epistemological triangle of "concept, sign and object" introduced by Steinbring (1994) (Fig. 4).

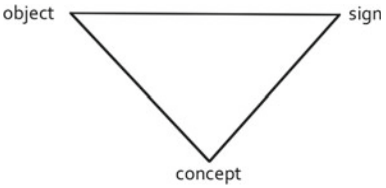

Fig. 4 The epistemological triangle

As a simple example think of the first natural numbers, denoted by the signs 1, 2, 3, ... and represented by sets of counters. In the early stages of the development of the number concept, counters are more meaningful and more operative than the signs. But the more the signs are filled with meaning and carry relationships, the more they contribute to better understanding operations with counters. At higher levels signs and connected symbolic operations can well be used as representations of new concepts and become accessible to experimental investigation.

By analyzing a number of lesson transcripts, Steinbring has shown that problems of understanding, ruptures, mistakes, misunderstandings etc. in the teaching/learning process are often due to treating or seeing the upper corners of the epistemological triangle in isolation or to subordinating one corner to the other one. In order to overcome the "socially conventionalized strictness of sign attribution" he argues in favor of "a conceptual flexibility for constructing references between symbol and referent" (p. 381).

3 Selection of Standard Number Representations

For instructional design the message resulting from the preceding analysis seems clear. In distinct opposition to the flood of "teaching aids" offered on the market manipulatives should be carefully selected: "Less is More".

The present section gives an account of how Mathe 2000 has approached the problem of selecting and designing a set of standard number representations for the teaching of arithmetic.

3.1 Criteria for Selecting and Designing Standard Representations

The following criteria for selecting and designing standard representations for a given domain of teaching have been developed and continuously modified in the process of design. On the one hand they re-phrase theoretical results of the preceding sections, on the other hand they reflect practical conditions and constraints of teaching.

Criterion 1:
The number of standard representations for teaching a given domain should be small so that the students can become thoroughly familiar with them in the time available for learning. In order to avoid ruptures in the learning process, these standard representations should be compatible with one another.
Criterion 2:
The standard representations should capture the fundamental mathematical ideas underlying the given domain as far as possible. This secures mathematical substance and opens up extended opportunities for students' structuring activities as the basis of mental images.
Criterion 3:
Standard representations should be available in two isomorphic user-friendly versions: a big one for the purpose of demonstration in the class, a small one for student's use. This facilitates the transition from individual and small group work to classroom communication and vice versa.

The big standard representations should be fixed on the walls of the classroom and be freely accessible, the small versions should be ready at hand.
Criterion 4:
Each student should be equipped with personal copies of the standard representations. Therefore the small versions must be made of low cost materials (as a rule paper).

Of course the leading criterion for selection and design is the second one. Its application depends crucially on the identification of fundamental ideas of the given domain, a task that is taken up in the following section for arithmetic.

3.2 Fundamental Ideas of Arithmetic

A developmental approach to teaching a given field of knowledge cannot be based on systematic-deductive presentations. What is needed instead is a genetic picture of this field. Here Bärbel Inhelder's suggestion to identify "fundamental ideas" which can be elaborated in the process and progress of learning has proved as a striking method (Bruner 1960, Chap. 2).

The following list of fundamental ideas of arithmetic has been based on the operative principle: In any domain of knowledge there are

- *"objects"*
- *"operations"* which can be applied to the objects and
- the *"effects"* of the operations on the properties and relationships of objects.

In arithmetic the "objects" are numbers, sums, differences, products, quotients, functions, etc. The "operations" are counting, adding, subtracting, multiplying, dividing, etc. The "effects" are expressed by the laws of arithmetic and all kinds of number patterns.

This operative structure is visible in the following list of fundamental ideas of arithmetic:

1. "Number as a synthesis of the ordinal and the cardinal aspect"
The natural numbers form an infinite series which is covered in counting (ordinal aspect). They also serve as cardinal numbers.

2. "Operating with numbers"
The laws of arithmetic provide the frame for (more or less sophisticated) mental and informal calculations as well as for algorithms. The laws of arithmetic are preserved in the larger domains (fractions, integers, real numbers).

3. "Decimal system"
Our conventional number system is based on the number ten. An important role is also played by the number 5 as a half ten ("Power of five", Flexer 1986). The thousand triade is repeated in the millions, milliards, etc.

4. "Standard algorithms"
Standard algorithms allow for reducing calculations with numbers to calculations with digits. The algorithms can be automatized and implemented on hand-held calculators and computers.

5. "Number patterns"
Arithmetic is rich in problems and number patterns (number theory, combinatorics).

6. "Numbers in the environment"
Natural numbers can be applied as cardinal numbers, ordinal numbers, magnitudes, operators and codes.

7. "Arithmetic as a language"
Real situations can be mathematised by using the conceptual structures of arithmetic.

3.3 Standard Number Representations

With the criteria of Sect. 3.1 and the list of fundamental ideas of Sect. 3.2 in mind, available manipulatives have been checked, selecting those which seemed most appropriate for the teaching units that were being developed.

It turned out very quickly that some of the traditional number representations were optimal solutions for our purposes (for example, the Hundred Table) and that others could be easily adapted (for example, the Hundred array). The remaining gaps were filled by newly developed materials (for example, posters for the addition and multiplication tables).

Some well-known and popular number representations did not meet our criteria and consequently had to be dropped. For example, the Cuisenaire rods, though offering a good potential for structuring activities in grade 1 and in part in grade 2, do not incorporate the "power of five" (criterion 2, fundamental idea 3), cannot be extended to grades 3 and 4 (criterion 1), and are too expensive to be available for every student (criterion 4). Moreover, when it comes to operating on sums, the Cuisenaire rods are less flexible than the Twenty frame and counters (criterion 2, fundamental idea 2). For similar reasons base ten blocks had to be excluded from the list.

The following manipulatives were chosen as standard number representations (cf. Wittmann and Müller 1990, 9–12, 1992, 10–12). They are listed here according to their correspondence to the fundamental ideas of arithmetic as they appear grade by grade[3]:

"Series of Numbers"

Grade 1:

- **Twenty Row** (Circles in groups of five, numbered from 1 to 20 or alternately with entries 5, 10, 15, 20)

Grade 2:

- **Hundred Row** (100 circles, coloured in groups of five, with entries 5, 10, 15, 20)

Grade 3:

- **Thousand Row** (yardstick-like representation of numbers from 1 to 1000 with entries 25, 50, 75, 100, 125, 150, . . .)

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[3] At this place a general comment seems appropriate. In German mathematics education there has been a steady work on teaching aids since the 19th century. This work has been accompanied by coining short and fitting names for teaching aids and a didactical language for their use. The German language has been facilitating this process by certain linguistic characteristics, for example, the easy formation of compounds.

When the present chapter and the textbook DAS ZAHLENBUCH were translated into English it has been impossible at some places to find suitable English terms. Therefore, the author has taken the freedom to coin new words that are more or less literal translations from the German. For English-speaking readers these terms may need habituation, and it is not unlikely that some might reject them together with the didactical context into which they belong.

It cannot be ignored that there is a general problem in translating meanings from one language and one cultural context into another one. In Hughes (1994, 311–312) the English poet Ted Hughes expresses his irritation when he realized that an urban American poet had degraded his poems. Hughes explains this lack of understanding with the spread of a lingua franca that is unavoidably developing in a multicultural society like the US. This second language has a tendency of becoming superficial and of ironing out deeper meanings, and it induces the speakers of this language to consider themselves as superior and to ignore what is communicated in other languages.

To some extent this problem also exists in mathematics education, and it becomes especially apparent at international conferences. The use of English as a lingua franca of mathematics education is a double-edged sword. Mathematics educators in non-English-speaking countries are well-advised to preserve and to cultivate their context and stick to proven achievements that are not reflected in the lingua franca.

Grade 4:

- **Number line** as a mental model indicated by representations of various sections of the number line.

  Hassler Whitney's idea of the *empty number line* (Treffers and de Moor 1990; Gravemeijer 1994, p. 120 ff.) has been integrated in grades 3 and 4. Figure 5 (from Treffers and de Moor 1990, 56–57), shows how this idea is used by children.

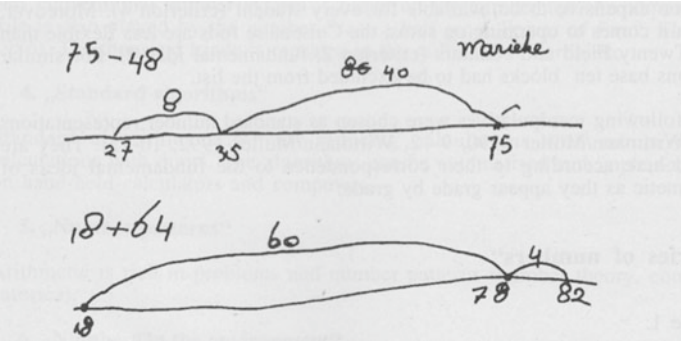


Fig. 5 Reasoning with the empty number line

"Calculating"

Grade 1:

- **Counters** (one side red, other side blue) for representing numbers, sums and differences as well as patterns
- **Number cards** for numbers 0 to 20 (one side: numbers written with digits, other side: corresponding pattern of dots) (Fig. 6)

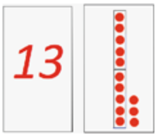


Fig. 6 Number cards

- **Twenty frame and counters** for a systematic study of the addition table (Fig. 7)

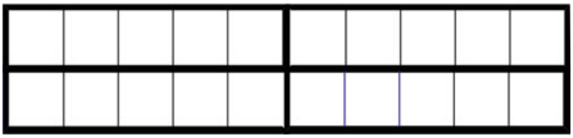


Fig. 7 Twenty frame

- **Addition chart** (big version 90 cm x 120 cm with colored boxes for demonstrating the operative structure of the addition table (Fig. 8).

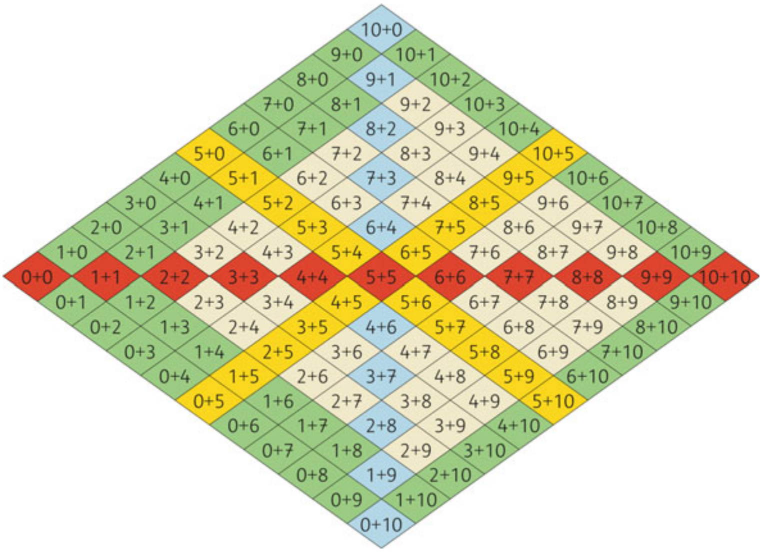


Fig. 8 Addition chart

Grade 2:

- **Hundred array** (subdivided into four 25 squares according to the "power of five") and "**cover card**" for showing numbers from 1 to 100 (Fig. 9).

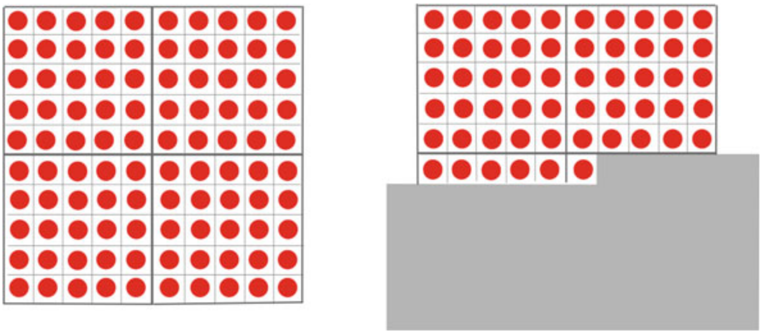


Fig. 9 Hundred chart with cover card

- **Bar/dot representation:** As a shorthand notation, bars are used for tens and dots for ones (Fig. 10).

Fig. 10 Bar/dot representation

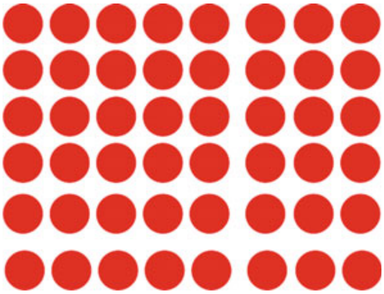


Fig. 11 Dot array

- **Dot arrays:** These arrays are used for representing products (Fig. 11).
- **Hundred array and "angle card"** for representing products (Fig. 12).

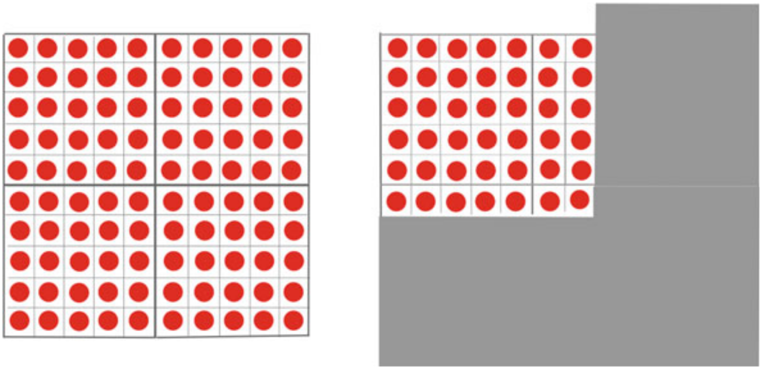
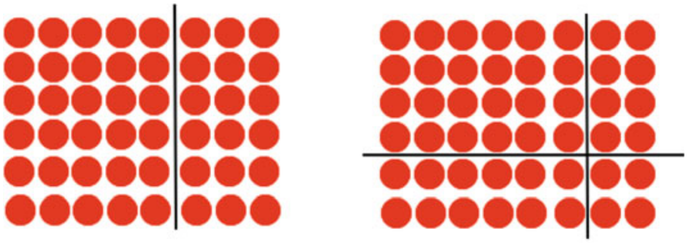


Fig. 12 Hundred array and angle card

- **Multiplication grid:** The subdivision of dot arrays by a line or a cross gives rise to a shorthand notation for computing products according to the distributive law (Fig. 13).

  The multiplication grid is very useful when it comes to calculating products of larger numbers.



·	5	3	
6	30	18	48

·	6	2	
4	24	8	32
2	12	4	16
			48

Fig. 13 Multiplication grid

- **Multiplication system** (big version 90 cm × 120 cm): Systematic overview of the multiplication table where the products are represented as strings of circles.
- **Multiplication chart** (big version 90 cm × 120 cm), structure analogous to the Addition chart (Fig. 14).

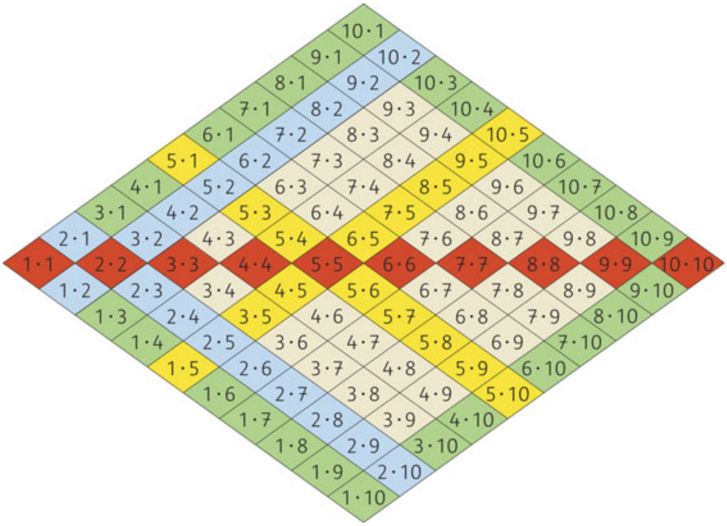


Fig. 14 Multiplication chart

Grade 3:

- **Thousand array** (Wittmann and Müller 1992, p. 1 0): 10 copies of the hundred array in linear order. This array is useful for representing numbers and for supporting calculations (Fig. 18).

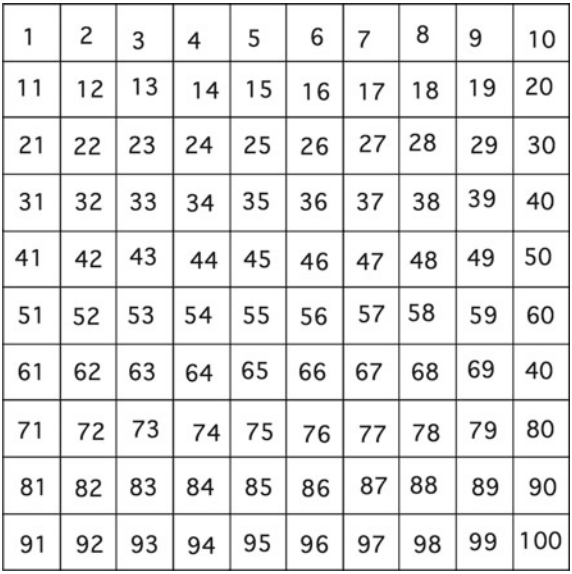
"Decimal System"

Grade 1:

- **Twenty frame** (Fig. 7)

Grade 2:

- **Hundred chart** (Fig. 15)



1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	40
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Fig. 15 Hundred chart

Grade 3:

- **Square/bar/dot representation of numbers** (Fig. 16)
- **Place value chart and counters** (Fig. 17)

Fig. 16 Square (bar/dot representation

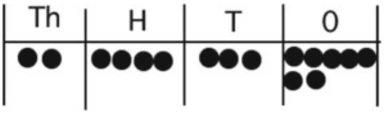


Fig. 17 Place value chart and counters

- **Thousand book** (Wittmann/Müller 1990, p. 10, 14ff.)

This teaching aid is a continuation of the Hundred chart. It reflects the triadic structure of the decimal system: 10 unit squares form a line, 10 lines form a page, and 10 pages form the whole book. Each page is isomorphic to the hundred chart (Fig. 18). The pages can be folded to make a Leporello (Fig. 18). The back of the Thousand book shows the Thousand array (Fig. 19)

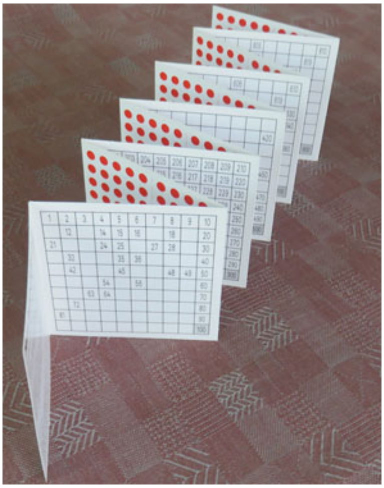


Fig. 18 Thousand book

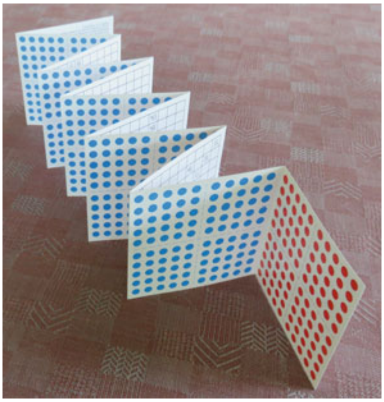


Fig. 19 Thousand array

Grade 4:

- **Place value chart and counters** with further columns at least up to the million
- **Million book:** The Thousand book if folded together is a square, like one unit field of the book itself. With this bigger square, the construction of the Thousand book is repeated on a higher level: 10 Thousand books (folded to squares) make one line (10 000), 10 lines make one page (100 000), and ten pages make the whole Million book. In this book every number from 1 to 1 000 000 has a definite place. For example, 365 278 is the 278th number in the 366th thousand book.

  The construction can be repeated infinitely: The Million book, when folded together is again a square, 10 squares make a line, 10 lines make a page, 10 pages make the Milliard book, and so forth.

  The triadic structure of the place value chart and the series of Thousand book, Million book, Milliard book, $\cdots$ reflect the triadic writing of natural numbers, for example 423 365 278. Within each triad the calculations are identical.
- **Digit cards for digits 0 to 9** (Fig. 20)

Fig. 20 Digit cards

"Standard Algorithms"

Grade 3:

- **Place value chart** for addition and subtraction

Grade 4:

- **Place value chart** for division
- **Napier's strips:** As a preliminary version of long multiplication a diagram used in the Middle Ages (written version of Napier's rods) can be easily derived from the Multiplication grid (Fig. 21).

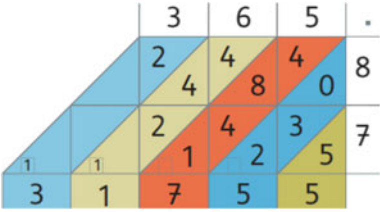


Fig. 21 Napier's strips

"Number Patterns"

All grades:

- **Patterns of counters, dot arrays, place value chart and counters**

"Numbers in the Environment" and "Arithmetic as a Language"

Over the grades:

- Yardsticks, square metre (made of paper), cubic metre (made of wooden rods), clocks, calendars, money, scales and weights, measuring vessels.

4 Some Teaching Units

The following brief sketches of teaching units developed in the project Mathe 2000 are to indicate how standard number representations can serve as tools for the learner in mathematizing, discovering, reasoning and communicating during the entire process of active and social learning.

It should become clear that the students enjoy all freedom in using the manipulatives, including the freedom not to use them, in other words, that standard representations do not necessarily involve standard ways of teaching and learning. On the contrary: If freed from a didactic system, standard representations, which by their very design are related to fundamental mathematical ideas, greatly contribute to a stable learning environment and stimulate the learning process in the same way as a rich and stable language environment in early childhood fosters the process of language acquisition. As convincingly shown by Dewey (1976) a learning environment carefully grounded on the subject matter of teaching is even a pre-condition for the success of open learning processes.

The following descriptions indicate the potential of the units which in real teaching is only rarely exhausted. The units can be used very flexibly and leave much room for natural differentiation. For further details the reader is referred to the "Handbook of Practicing Skills in Arithmetic" (Wittmann/ Müller 1990, 1992).

4.1 The Twenty Frame and the Addition Table (Grade 1)

In opposition to the traditional method the (open) number space 1 to 20 is introduced as a whole at the beginning of grade 1 fairly quickly and considered in several rounds from different sides. Similarly, the addition table is studied in a holistic way. After considering "additive situations" in the environment and eliciting children's spontaneous addition strategies, a first systematic study of the addition table is built upon the Twenty frame. Children can represent and solve addition tasks in different ways. In principle there are no prescribed methods. It is only by trying different placements and groupings of counters, by changing counters and by discussions in small groups or in the class that the individual child will discover relationships and learn to use them according to his or her personal preferences in mastering the addition table. Of course the "power of five" will be experienced by all children as a very useful strategy which, however, is to be used flexibly. The Twenty frame supports this strategy.

For example, let us consider the task $7 + 5$.

Representation A (Fig. 22): 7 red counters are placed in the first row, 5 blue ones in the second one. The two fives on the left side are grouped together to make 10.

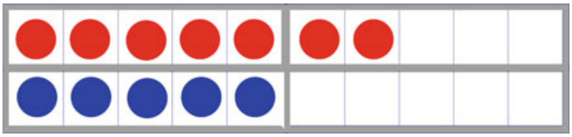


Fig. 22 First solution of $7 + 5$ with the twenty frame

Representation B (Fig. 23): Again 7 red counters are placed in the first row, 3 of 5 blue counters fill the first row and leave two counters for the second row. This is the traditional "transition beyond ten".

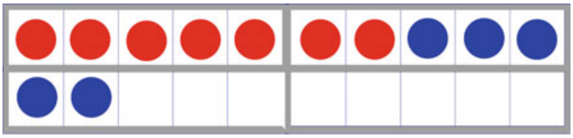


Fig. 23 Second solution of $7 + 5$ with the twenty frame

Representation C (Fig. 24): One red counter is translated to the second row. Before calculating the task $7 + 5$ is changed into $6 + 6$, a task with the same result which, however, is easier for many children.

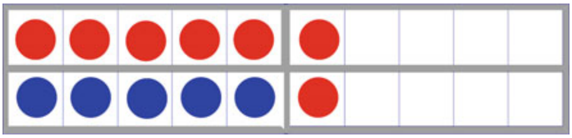


Fig. 24 Third solution of $7 + 5$ with the twenty frame

It should be mentioned that this approach to the addition table via the Twenty frame is essentially identical with the approach suggested by Treffers via the arithmetic rack (cf., Gravemeijer 1994, 71–72).

4.2 *Multiplication Chart (Grade 2)*

This poster is used for a systematic study of the operative relationships between multiplication tasks. From the rich source of activities, a unit for practicing multiplication facts combined with discovery and proof is chosen.

The students are asked to calculate the following pairs of tasks on the Multiplication chart:

$$\begin{array}{llllll} 1 \cdot 1 = \quad, & 2 \cdot 2 = \quad, & 3 \cdot 3 = \quad, & 4 \cdot 4 = \quad, & 5 \cdot 5 = \quad & \ldots \; 10 \cdot 10 = \quad . \\ & 1 \cdot 3 = \quad, & 2 \cdot 4 = \quad, & 3 \cdot 5 = \quad, & 4 \cdot 6 = \quad & \ldots \end{array}$$

Most children will find out that the results of each vertical pair differ by 1. Some will discover the missing partners of the first and the last task in the first row ($0 \cdot 2$ and $9 \cdot 11$) and state that the difference is again 1.

By means of counters children can be guided to look at special cases and to find an operative proof *why* the difference *must* be 1 (Fig. 25). For example, the pattern of $5 \cdot 5$ has one row, that is 5 counters, more than the pattern of $4 \cdot 6$. The latter has one column, that is 4 counters, more than the former. Therefore $5 \cdot 5$ must be 1 more than $4 \cdot 6$. The same operations can be applied to other patterns. Stimulated by this operative series, students can investigate other adjacent pairs of rows or columns on their own and look for similar relationships.

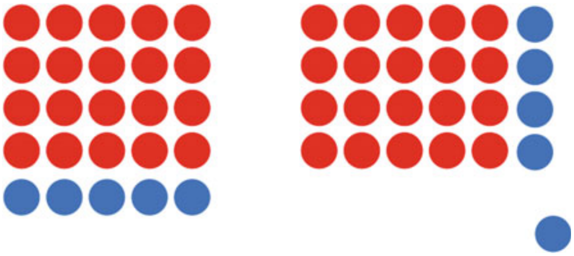


Fig. 25 Number pattern

4.3 An Introduction into the Thousand Book (Grade 3)

“Introducing” the Thousand book means to have the children explore its structure without predetermining objectives. At the beginning of the unit, each child gets a personal copy of the book and is stimulated to think about it. After some time the children are asked to report on their ideas. For the teacher this “local finding” yields valuable information for further teaching.

If the question “Why is the Thousand book called a’book’?” is not asked spontaneously by the children, the teacher has to pose it and to make sure that the students become aware of the “lines” and “pages”.

A good stimulation for a deeper exploration is the following suggestion: “Find tasks with the result 1000 and write them down on a sheet of paper!” This open problem allows for “natural differentiation”. Some children stick to simple problems like $500 + 500, 600 + 400, 999 + 1$. Others find tasks like $10 \cdot 100, 50 \cdot 20$ or $40 \cdot 25$ and go beyond 1000, for example $10000 - 9000 = 1000$.

Another good problem for studying the back of the Thousand book, the Thousand array, is to divide 1000 (“smarties”) among 2, 3, 4, . . ., 10 children.

This introductory unit shows very clearly that the structure of the number space 1 to 1000 is not “taught” in the traditional sense. It is “learned” through exploring problems related to its structure. During the exploration there is plenty of room for students’ own initiative and choice.

4.4 “Always 22” (Grade 3)

This unit is based upon the following rule:

(1) Select three digit cards from the nine cards $1, \ldots, 9$ (for example 2, 4, 7)
(2) Form all possible two-digit numbers (24, 27, 42, 47, 72, 74) and add them

$$(24 + 27 + 42 + 47 + 72 + 74 = 286)$$

(3) Divide this sum by the sum of the three selected digits

$$(2 + 4 + 7 = 13, \quad 286 : 13 = 22)$$

There are many (84) triples of digits, and different children will make different choices. The more surprising is the fact that independently of the selected digits the result of the division task is always the number 22 (giving the unit its name)—provided the calculations are correct.

For one case the explanation of this observation is easy, namely when the sum of digits is exactly 10 (for example for the digits 2, 3, 5). Representing the numbers in the place value chart shows how the sum of digits is related to the sum of numbers: The sum of digits appears twice in the Ones column and twice in the Tens column. Division by the sum of digits necessarily yields 2 tens and 2 ones, that is 22. The same patterns occurs with other triples.

Again we have an example where standard representations are powerful enough to establish a proof.

4.5 Place Value Chart (Grade 4)

A good activity for the operative study of the place value chart is provided by the following problem: Which numbers can be represented by 3 (or 2, 1) counters on a place value chart with four columns?

Children can explore this problem more or less systematically. It is not necessary that all children discover all 20 possible numbers. However, if the children are told that there are in all 20 numbers, there is a good chance that the class as a social body will find them all. It is a nice activity to order them (and to determine the differences of adjacent numbers, particularly if the problems with 2 counters and 1 counter have been solved previously).

In the Mathe 2000 curriculum this combinatorial problem does not come out of the blue but is part of a strand of combinatorics running from grade 1 to 4. In grade 3 the present problem is prepared by the corresponding problem for the place value chart with *three* columns.

In grade 1 the children investigate how many different Easter nests can be found with 3 (4) eggs where for the eggs the colors red, blue and yellow are available. These problems are structurally isomorphic to the problems with the place value chart.

In grade 2, 3 and 4 the children determine the number of different shortest ways in a grid from one vertex to another one with a certain distance (measured in the grid metric).

5 Conclusion

The design of a curriculum is only half of the way to reshaping mathematics teaching. In order to get deeper insights into the teaching and learning of mathematics empirical research on a large scale is needed. The research findings will certainly give rise to re-working, modifying and refining the design. In this sense design depends very much on empirical research. But the reverse is equally true. Good design is also a pre-condition for productive empirical research as rich learning environments are much more likely to yield substantial results.

The author is convinced that because of this interdependence of design and empirical research a systematic cooperation between instructional designers and empirical researchers, grounded on a solid theoretical basis, will open a new chapter in the development of mathematics education as a discipline.

Acknowledgements The author is indebted to Adrian J. Pinel, Wilhelm Schipper and Adri Treffers for helpful comments.

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Chapter 9
Developing Mathematics Education in a Systemic Process

The aim of this paper is to make a concrete proposal for bridging the gap between theory and practice in mathematics education and for establishing a systemic relationship between researchers and teachers as well as to explain the background and the implications of this proposal.[1]

1 Bridging the Gap Between Theory and Practice: The Role of Substantial Learning Environments

> You cannot fail if you follow the advice the genius of human reason whispers in the ear of each new-born child, namely to test thinking by doing and doing by thinking.
>
> J.W. von Goethe

In Guy Brousseau's book "Theory of Didactical Situations in Mathematics" the scene is set with a teaching example, the "race to 20", which is based on a game of strategy (Brousseau 1997, 3–18). In a somewhat modified version this game can be described as follows (cf. Fig. 1).

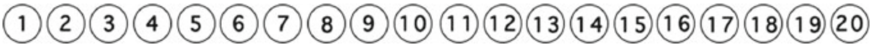

Fig. 1 Plan for the game "The race to 20"

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[1] Plenary Lecture at the 9th International Congress on Mathematical Education (ICME 9) Makuhari/Japan, July 31–August 6, 2000. The paper was prepared during a sabbatical leave at the University College Chichester/England. The author is indebted to Afzal Ahmed, Brian Griffiths, Honor Williams and Heinz Steinbring for critical comments. The author also would like to express his gratitude to Jerry P. Becker who was one of the very first Western mathematics educators to recognise the impact of research conducted by Asian mathematics educators. Since the seventies Jerry Becker has been providing first hand information about the Japanese approach to mathematics education which has greatly influenced the author's research.

E. C. Wittmann, *Connecting Mathematics and Mathematics Education*,
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A line of circles is numbered from 1 to 20. The first player starts by putting 1 or 2 counters on the first circle or the first two circles, the second player follows by putting 1 or 2 counters on the next circles similarly. Continuing in this way the players take turns until one of them arrives at 20 and in doing so wins the game.

The "race to 20" helps to corroborate basic arithmetical ideas (relationships of numbers on the number line, addition, repeated addition). It is also a rich context for general objectives of mathematics education (exploring, reasoning and communicating) and a typical example of the fundamental principle of "learning by inquiry". If children analyse the moves backwards they recognise that the positions 17, 14, 11, 8, 5 and 2 are winning positions. So the first player has a winning strategy: In the first move she puts down two counters and then responds to a 2-counters move of the second player with a 1-counter move and to a 1-counter move of her opponent with a 2-counters move. In this way the first player jumps from one winning position to the next one and finally arrives at 20.

There are many variations of this game: Any natural number can be chosen as the target, and the maximal number of counters to be put down on every move can be increased. In fact we have a whole class of games of strategy before us which require a continuous adaptation of the strategies used.

The basic ideas of analysing these games can be generalised to the wider class of finite deterministic games of strategy for two persons with full information which cannot end in a draw: by means of the game tree and the marking algorithm one can prove that for each of these games there exists a winning strategy either for the first or the second player.

As mentioned in Brousseau's book the "race to 20" was reproduced 60(!) times under observation and each of its phases was the object of experimentation and clinical study. Based on a variety of other teaching examples Brousseau developed his theory of didactical situations. In the research context "aspects of proving" Galbraith (1981) studied students' psychological processes in their attempts to uncover the structure underlying the "race to 25."

The "race to 20" and its variations represent what has been called a substantial learning environment, an SLE, that is a teaching/learning unit with the following properties (Wittmann 1995, 365/366):

(1) It represents central objectives, contents and principles of teaching mathematics at a certain level.
(2) It is related to significant mathematical contents, processes and procedures *beyond* this level, and is a rich source of mathematical activities.
(3) It is flexible and can be adapted to the special conditions of a classroom.
(4) It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research.

The concept of an SLE is a very powerful one. It can be used to tackle successfully one of the big issues of mathematics education which has become more and more urgent and which is of crucial importance for the future of mathematics education as a discipline: the issue of theory and practice. Fortunately, for some years now this issue has been more and more recognised and addressed by mathematics educators.

Referring to the recent ICMI Study "Mathematics Education as a Research Domain" (Sierpinska and Kilpatrick 1998) Ruthven stated that there is a wide gap between the scholarly knowledge of researchers on the one hand and the craft knowledge of teachers on the other hand, and argued in favour of a re-orientation of mathematics education:

> While most of the contributors identify the development of knowledge and re-sources capable of supporting the teaching and learning of mathematics as an important goal for the field, there is disappointment with what has been demonstrated on this score.
>
> (Ruthven, 2001)

A claim similar to Ruthven's was made by Clements and Ellerton for the South East Asian and by Stigler and Hiebert for the American context:

> From our perspective, at the present time mathematics education needs less theory-driven research, and more reflective, more culture-sensitive, and more practice-orientated research which will assist in the generation of more domain-specific theory.
>
> (Clements and Ellerton 1996, 184)

> Perhaps what teachers are told by researchers to do makes little sense in the context of an actual classroom. Researchers might be very smart. But they do not have access to the same information that teachers have as they confront real students in the context of real lessons with real learning goals . . . It is clear that we need a research-and-development system for the steady, continuous improvement; such a system does not exist today.
>
> (Stigler and Hiebert 1999, 126–127)

This criticism can be extended: for teachers' decision-making the logical and epistemological structure of the subject matter is at least as important as are psychological, social or more general aspects of learning and teaching. However, in the mainstream of current research in mathematics education this very structure has not received the attention it deserves. Therefore the gap between theory and practice is also due to a gap between mathematics on one side and mathematics education on the other side. This gap is particularly obstructive to progress in reforming mathematical education as the epistemological structure of the subject matter contains psychological and social aspects at least implicitly while the converse does not hold.

Of course the issue of relating theory and practice to one another is not specific for mathematics education. In all fields we have, at one extreme, mere "doers" who act in a pragmatic manner, who don't see any point in worrying about theory and who even think of theory as a threat to practice. At the other extreme are mere "thinkers" who develop analyses and theories with no grounding in practice and without caring for practical implications and applications.

In tackling the issue of theory and practice a superficial re-arrangement of the field is not sufficient. If we seriously want to establish links between theory and practice a fundamental change is needed. For systemic reasons it is highly unlikely that theories which have been developed independently of practice can be applied afterwards:

> The developing theory of mathematical learning and teaching must be a refinement, an extension and a deepening of practitioner knowledge, not a separate growth

as stated by Alan Bell in the mid-eighties (Bell 1984, 109).

Therefore in order to organise a strong and lasting systemic interaction between theorists and practitioners we have to look for some common core in which theory and practice as well as mathematics and mathematics education inseparably permeate one another. Substantial learning environments can serve this purpose quite naturally (cf., Wittmann 1984; Wittmann 1995/1998). Accordingly, the main proposal of this paper is as follows (see Fig. 2 which is an extension of a diagram presented in Ruthven 2000):

The design of substantial learning environments around long-term curricular strands should be placed at the very centre of mathematics education. Research, development and teacher education should be consciously related to them in a systematic way.

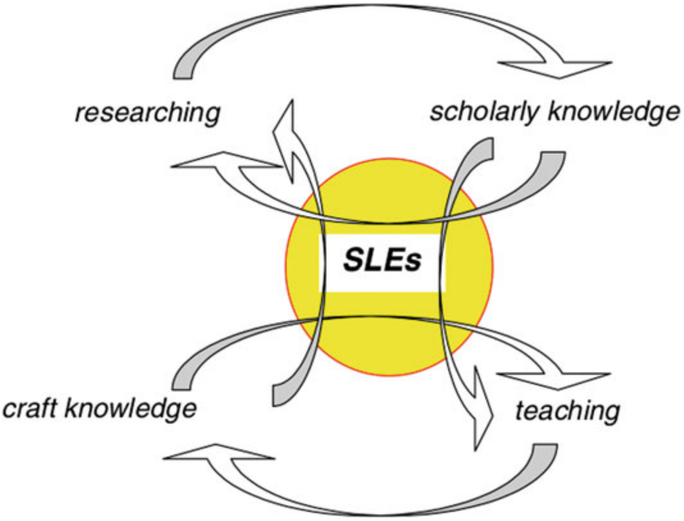


Fig. 2 The role of substantial learning units in connecting the teaching practice with research

This proposal is supported by encouraging experiences which have been made in various projects around the world. Prominent examples are the work of the British Association of Teachers of Mathematics in the sixties (cf., Fletcher 1965; Wheeler 1967), the prolific Dutch *Wiskobas* project and its follow-up projects conducted at the Freudenthal Institute, and the systematic work of Japanese mathematics educators (cf., Shimada and Becker, 1996). These projects show what an important role SLEs can play for both researchers and practitioners: as common points of reference, as knots in the collective memory, and as stimuli for action. The proposal reflects a certain understanding of the particular nature of the system of education which will be examined in the next section.

2 (Burst) Dreams

> Variety can only be absorbed by variety.
>
> Ross Ashby

It is not by chance that development projects based on SLEs have been successful in changing mathematics teaching as well as in changing teachers' attitudes: in these projects fundamental systemic conditions have been taken into account. This will be explained in more detail by referring to three dreams that were dreamt by a prominent philosopher, a prominent mathematician and a prominent educator. These dreams have been selected because they capture the non-systemic tradition of teaching and learning which must be fully recognised in order to be overcome.

2.1 Descartes' Dream

In 1619 the young René Descartes (1596–1650) had a vision of the "foundations of a marvellous science", on which he elaborated later in several writings, particularly in his *Discourse on the method of properly guiding the reason in the search of truth in the sciences* (Descartes 1637). Basically this method consisted of a few rules by which the mind can arrive at more and more complete descriptions of reality. In modern words the method was a totalitarian programme for mathematising reality. By separating the thinking mind, the *res cogitans*, from the world outside, the *res extensa*, Descartes established a sharp split between man and his environment which later on became a fundamental ideology of Western thinking. Already before Descartes Francis Bacon (1561–1626) had formulated the inductive method of science and summarised its technological use in the slogan "Knowledge is Power". So from the very beginning Descartes' dream of arriving at a complete description of the environment was accompanied by the dream of controlling and making use of it. The "Cartesian system" of philosophy, as it was called later, has paved the way for an unrestrained mathematisation, control and also exploitation of more and more parts of our natural and social environment. In our time the availability of computers has accelerated this process (cf. Davis and Hersh 1988). "Benchmarking", "controlling", "evaluation", and "assessment" have become key notions in the management hierarchies of economics and administration.

2.2 Hilbert's Dream

At the turn to the 20th century the very science in which Descartes wanted to ground truth, mathematics, was fundamentally shaken by the discovery of inconsistencies within Cantor's set theory. Among those mathematicians who were particularly alarmed was David Hilbert. In order to defend "the paradise", which, in his eyes, Cantor had created, he started the so called “finitistic programme” by which he hoped to prove the consistency and infallibility of mathematical theories once and for all (Hilbert 1926). Although Hilbert’s dream burst already in 1930 when Gödel proved his incompleteness theorem, the formalistic setting of Hilbert’s programme has survived and turned into an implicit theory of teaching and learning. Interestingly, the Bourbaki movement which set formal standards in mathematics up to the seventies started in the mid-thirties from a discussion about how to teach analysis. Also in this day and age the belief in formal precision as a necessary if not sufficient means to manage teaching/learning processes is still widespread among mathematicians and non-mathematicians. The *Mathematically Correct* movement, at present one of the most aggressive pressure groups in the U.S., is a horrifying example.

2.3 Comenius’ Dream

Johann Amos Comenius (1592–1670) is well-known as one of the founding fathers of didactics. His famous book “Great Didactic” published in 1657 was the first comprehensive work on teaching and learning. In many respects Comenius was far ahead of his time. For example, he was among the first to project a plan of universal education and to see the significance of education as an agency of international understanding. In one respect, however, he was a child of his time. Deeply impressed by Bacon’s visions of a technological age and by the efficiency of newly invented machines, he was obsessed by the idea of transposing the functioning of machines to the functioning of teaching. In the Chaps. 13 and 32 of the “Great Didactic” he states:

> The art of teaching, therefore, demands nothing more than the skilful arrangement of time, of the subjects taught, and of method. As soon as we have succeeded in finding the proper method it will be no harder to teach school-boys, in any number desired, than with the help of the printing-press to cover a thousand sheets daily with the neatest writing … The whole process will be as free from friction as is the movement of a clock whose motive power is supplied by the weights. It will be as pleasant to see education carried out on my plan as to look at an automatic machine of this kind, and the process will be as free from failure as are these mechanical contrivances, when skilfully made … Knowledge can be impressed on the mind, in the same way that its concrete form can be printed on paper. (Comenius 1910, 96–97, 289)

Comenius’ dream has been dreamt over the centuries in ever new forms and is still present in some corners of cognitive science and education, including mathematics education, as various forms of “direct teaching” and “hard science”-like methods of research demonstrate (cf., for example, Begle 1979).

Also a certain tendency within the research community to consider teachers as mere recipients of research results is clearly related to Comenius’ dream:

> I suspect that if teachers are mainly channels of reception and transmission, the conclusions of research will be badly deflected and distorted before they get into the mind of pupils. I am inclined to believe that this state of affairs is a chief cause for the tendency . . . to convert

> scientific findings into recipes to be followed. The human desire to be an "authority" and to control the action of others does not, alas, disappear when a man becomes a researcher. (Dewey, 1929/1988, 24)

2.4 The 'Systemic-Evolutionary" Versus the "Mechanistic-Technomorph" Approach to the Management of Complexity

It may seem as too far-fetched to look at Descartes, Hilbert and Comenius from the point of view of modern systems and management theory. However, there is a good reason to do so, for the three dreams, as different as they may appear, share a common feature: They reflect the self-concept of individuals who perceive themselves as standing on a higher level and as equipped with the capacity to gather complete information about some field and to use this information for bringing this field under control. The Swiss management theorist Malik has called this attitude the "mechanistic-technomorph approach to the management of complexity" and described it as follows:

> The paradigm [underlying this approach] is the machine in the sense of classical mechanics. Basically, a machine is constructed according to a given purpose and to a given plan, and its function, reliability and efficiency depend on the functions and the properties of its elementary components . . . The technological success which has been achieved by following this paradigm is overwhelming, and gave rise to the belief in its unlimited applicability far beyond the engineering disciplines. . . . The paradigm includes the firm conviction that no order whatsoever which corresponds to human purposes can be brought about without following this paradigm. (Malik 1986, 36 ff., transl. E.Ch.W.)

During the past decades another paradigm has been taking ground, based on the fact that biological and social organisms are far too complex in order to allow for a "mechanistic-technomorph" description and control from outside. In order to achieve certain goals with living systems a fundamentally different approach is appropriate:

> The systemic-evolutionary approach [to the management of complexity] starts from quite different assumptions. Its basic paradigm is the spontaneous, self-generating ordering exemplified best by the living organism. Organisms are not constructed, they develop. Spontaneous orderings develop also in the social domain. They arise by means of and as the result of human actions, but they do not necessarily correspond to preconceived intentions, plans or goals. Nevertheless they can be highly rational. (Malik 1986, 38 ff, transl. E.Ch.W.)

According to the systemic-evolutionary paradigm the only reasonable and feasible way of influencing and guiding a social system is to interact sensibly with the self-organising powers inside the system. Recommendations and instructions from outside which do not fit into the internal processes of the system are, at best, useless. If, in addition, a minute control is exerted from the outside, the development of spontaneous powers inside the system is suppressed, and this undermines its efficiency. A system without a proper infrastructure is not able to interact adequately with a complex environment: variety can only be absorbed by variety.

The systemic-evolutionary approach to the management of complexity has been developing in Western philosophy only during the last decades. So it is even more astounding that it has emerged in Asia more than 2000 years earlier when Lao Tzu and Chuang Tzu founded the philosophy of taoism. The basic maxim of taoism for leaders is “wu wei”. This means: leaders should not interfere with the natural powers and inclinations of their clients, but should instead build upon self-organisation and offer help for self-help. It is the present author’s hope that the Asian societies will succeed in preserving the systemic-evolutionary thinking as a precious heritage from their past while it is spreading only slowly and with great difficulties in Western societies which are still in the claws of deeply rooted mechanistic-technomorph patterns of thinking and action.

3 Consequences for Mathematics Education

> A little child needs no famous teacher to learn to speak. He or she learns to speak spontaneously in the company of people who can speak.
>
> Chuang Tzu

Individual students, individual teachers, classrooms, staffs, school districts, states, countries: all are living organisms and therefore highly complex systems. *Beyond any political or educational ideologies* the following systemic conclusions can be drawn just from the natural law of the inherent complexity of these systems:

1. Learning unfolds best if the spontaneous powers of all involved are brought to bear and encouraged, and if autonomy and self-responsibility are developed.

The inevitable results of—possibly well-intended—straitjacket schemes of teaching, assessment and accountability are “over-standardisation, over-simplification, over-reliance on statistics, student boredom, increased numbers of dropouts, a sacrifice of personal understanding and, probably, a diminution of diversity in intellectual development.” (Stake 1995a, 213).

2. The traditional borderline between the researcher on one side and the teacher on the other side has to be abandoned. Research has to build upon the spontaneous powers of teachers in the same way as teaching has to build upon the spontaneous powers of students.

Donald Schön described this new relationship between theorists and practitioners very convincingly in his book *The Reflective Practitioner* (Schön 1983, 323):

> In the kinds of reflective research I have outlined, researchers and practitioners enter into modes of collaboration very different from the forms of exchange envisaged under the model of applied science. The practitioner does not function here as a mere user of the researcher’s product. He reveals to the reflective researcher the ways of thinking that he brings to his practice, and draws on reflective research as an aid to his own reflection-in-action.

3. At all levels the traditional hierarchies have to be transformed into networks of co-operation and mutual support.

A good account of what this means in different contexts is given in Burton 1999.

Although the "mechanistic-technomorph" paradigm of management is still dominant in all fields of society around the world, the awareness of the systemic nature of teaching and learning is steadily growing. As far as research and development in mathematics education are concerned there are already innovative research programmes which follow the new paradigm with remarkable success, for example *developmental research* (Freudenthal 1991; Gravemeijer 1994), Guy Brousseau's *theory of didactical situations* (Brousseau 1997), the Japanese *lesson studies* (Stigler and Hiebert 1999, Chap. 7), and *action research* (cf. Ahmed and Williams 1992; Clements and Ellerton 1996, Chap. 5).

The impact of these research programmes on practice rests on the fact that they are systematically focused on the design and empirical research of SLEs. A firm basis for a systemic researcher-teacher-interaction for "SLE studies" is thus provided, as illustrated by the following examples.

Example 1 In German primary schools the traditional approach to arithmetic in grade 1 has been to introduce the number space 1–20 step by step: The first quarter of the school year is restricted to the numbers 1–6, the second quarter to the numbers 1 to 10. The third quarter is open to numbers 1 to 20, however, tasks like $7 + 5$, in which the 10 has to be bridged, are postponed to the last quarter of the school year. Moreover, children are expected to follow the arithmetic procedures given by the teacher.

In the project Mathe 2000 this traditional approach was challenged and replaced by a holistic approach: The open number space 1 to 20 is introduced fairly quickly as one whole, children are encouraged to start from their own strategies and are not restricted to just one procedure. This new approach was formulated and published as a connected series of SLEs in a handbook for practising skills (Wittmann and Müller 1990; Grade 1: Chaps. 1–3). It was based on a systematic epistemological analysis of arithmetic, on inspirations from the developmental research conducted at the Freudenthal Institute (cf., Treffers et al. 1989/1990; Van den Heuvel-Panhuizen 1996) and on the intuitions of the designers. It was not based on empirical research conducted by professional researchers. Empirical studies, which confirmed the holistic approach, came only later (cf. Selter 1995; Hengartner 1999). Thus teachers were the first to try it out in their practice and they found that it works better than the traditional approach. Through the existing networks of teachers this new approach has spread widely in a remarkably short period of time. An innovative textbook is presently available (Wittmann and Müller 2000), based on the holistic approach, and its wide acceptance by teachers has convinced authors of traditional textbooks to modify their approach.

From the systemic point of view the success of this innovation is not surprising:

> One might ask the general question whether, in the present state of our knowledge about mathematical education, we should progress faster by collecting "hard" data on small questions, or "soft" data on major questions. It seems to me that only results related to fairly important practitioner questions are likely to become part of an intelligent scheme of knowledge … Specific results unrelated to major themes do not become part of communal knowledge. On

> the other hand, "soft" results on major themes, if they seem interesting and provocative to practitioners, get tested in the myriad of tiny experiments which teachers perform every day when they "try something and see if it works". (Bell 1984, 109)

Example 2 The second example is an SLE from the Japanese "open-ended approach". This unit was thoroughly researched before it was published (Hashimoto 1986; Hashimoto and Becker 1999). Its guiding problem is the so-called "matchstick problem": Children are shown a linear arrangement of squares (Fig. 3) and asked to find out how many matchsticks are needed to build 5, 6, 7 or more squares.

Fig. 3 The matchstick problem

There is a great variety of counting strategies to solve this problem. After having discussed the various solutions, children determine the number of matchsticks needed for other numbers of square, and try to find a general formula. In a similar way arrangements of matchsticks with more rows can be studied. Based on these concrete examples fundamental counting principles of combinatorics can be extracted, for example the addition principle and the principle of multiple counting (cf., Schrage 1994).

Systematic lesson studies of the matchstick problem provided exactly the professional knowledge teachers need in order to teach this unit successfully. The matchstick problem has then been included in a textbook (Seki et al. 1997, 117–118).

Stigler/Hiebert comment on the impact of lesson studies as follows:

> The knowledge contained in these reports … is not made up of principles devoid of specific examples or examples without principles. It is theories linked with examples. This knowledge is notable in several respects. First, theoretical insights are always linked with specific referents in the classroom. When a lesson-study group reports, for example, that one of its hypotheses has been supported, it is never outside the context of a specific lesson with specific goals, materials, students, and so on, all of which would be described in the report. (Stigler and Hiebert 1999, 163)

Example 3 A third example is provided by Heinz Steinbring's empirical research on the interplay between the epistemological structure of the subject matter and psychological and social factors (cf., for example, Steinbring 1997). Although highly theoretical, his research is strongly related to SLEs which are part of the current teaching practice. So the applicability of the research results is guaranteed from the very outset.

4 Substantial Learning Environments for Practising Skills

> What counts is not memorising, but understanding, not watching, but searching, not receiving, but seizing, not learning, but practising.
>
> A. Diesterweg

Focusing mathematics education on substantial learning environments involves the risk that must be clearly recognised in order to be avoided: substantial mathematics is fundamentally related to mathematical processes such as mathematising, exploring, reasoning and communicating. These are *higher order thinking skills*. Emphasising them can easily lead to neglecting basic skills, in particular at a time when efficient calculators and computers are available. Basic skills also tend to be neglected for another reason: to a large extent traditional ways of teaching mathematics consisted of prescribed procedures and their stereotyped practice. In their eagerness to get rid of “teach them and drill them”routines in favour of “constructivist” ways of learning and teaching reformers easily get trapped: they tend to identify practice with stereotyped practice, and by abolishing stereotypes they are likely to do away with the practice of skills at all.

As the mastery of basic skills is an indispensable element of mathematical competence we have to find ways how to integrate the practice of skills into substantial mathematical activities. This is not an easy task, as stated, for example, by Ken Ross:

> … drills of important algorithms that enable students to master a topic, while at the same time learning the reasoning behind them, can be used to great advantage by a knowledgeable teacher. Creative examples that probe students’ understanding are difficult to develop but are essential. (Ross 1998, p. 253)

The following example of an SLE (cf. Wittmann and Müller 1990, grade 2, Chap. 3.3) illustrates how the practice of a basic skills and the development of higher order skills can be combined. The example refers to an area of arithmetic which is notorious for drill and practice: the multiplication table.

The epistemological structure of the unit is unfolded in a heuristic manner as this is the best way to capture the potential of an SLE for both teaching and research (see also Sect. 5.1).

The rule on which the unit is based is very simple: With two arbitrarily chosen pairs of consecutive numbers two calculations are performed: one “top down,” the other one “crosswise” (Fig. 4).

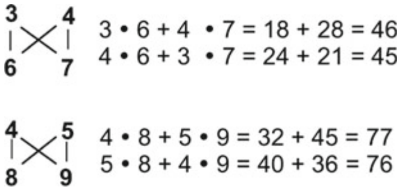


Fig. 4 Multiplying “top down” and “crosswise”

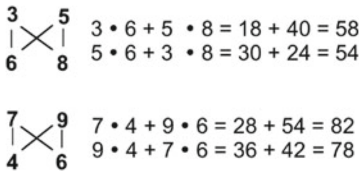
After sufficiently many calculations with numbers chosen by the children themselves a pattern is recognised: The result obtained “top down” seems always ‘1 bigger than the result obtained “crosswise”. Children who have found pairs for which this relationship does not hold will spot some mistake in their calculations.

In trying to explain the pattern children have to go back to the meaning of multiplication: $3 \cdot 6$ means $6 + 6 + 6$, $4 \cdot 7$ means $7 + 7 + 7 + 7$, etc. So $3 \cdot 6 + 4 \cdot 7$ contains one 7 more and one 6 less than $3 \cdot 7 + 4 \cdot 6$ which gives it an advantage of 1 (Fig. 5).

$$3 \bullet 6 + 4 \bullet 7 = 6 + 6 + 6 + 7 + 7 + 7 + 7 = 18 + 7 + 21$$
$$4 \bullet 6 + 3 \bullet 7 = 6 + 6 + 6 + 6 + 7 + 7 + 7 = 18 + 6 + 21$$

Fig. 5 Comparing the results

Of course the standard proof of this relationship is an algebraic one employing variables which are not available in grade 2. But variables are not needed at this level, the argument used above is absolutely appropriate. As a next step the distributive law can be made more explicit. For example, $3 \cdot 6 + 4 \cdot 7$ can be written as $3 \cdot 6 + 3 \cdot 7 + 7$ and compared with $3 \cdot 7 + 4 \cdot 6$ written as $3 \cdot 7 + 3 \cdot 6 + 6$. This pre-algebraic form is an excellent preparation for algebra in higher grades. As is typical for substantial learning environments the activity can be extended: Instead of pairs of consecutive numbers pairs of numbers which differ by 2, 3 or any other number can be chosen. In Fig. 6 the differences are 2.

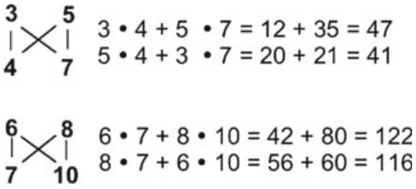


$$3 \bullet 6 + 5 \bullet 8 = 18 + 40 = 58$$
$$5 \bullet 6 + 3 \bullet 8 = 30 + 24 = 54$$

$$7 \bullet 4 + 9 \bullet 6 = 28 + 54 = 82$$
$$9 \bullet 4 + 7 \bullet 6 = 36 + 42 = 78$$

Fig. 6 First modification of the problem

The differences can also be mixed (Fig. 7).



$$3 \bullet 4 + 5 \bullet 7 = 12 + 35 = 47$$
$$5 \bullet 4 + 3 \bullet 7 = 20 + 21 = 41$$

$$6 \bullet 7 + 8 \bullet 10 = 42 + 80 = 122$$
$$8 \bullet 7 + 6 \bullet 10 = 56 + 60 = 116$$

Fig. 7 Second modification of the problem

From these examples a general pattern is emerging: the difference of the results of the two calculations is the product of the differences of the given numbers. It is not difficult to generalise the above proof for the introductory case. All one has to do is to decompose the second product in both calculations according to the distributive law.

Furthermore: beyond pairs of numbers triples of consecutive numbers (Fig. 8) and triples with fixed differences can be considered. In this case the "top down" result can be compared with two other results: one obtained by multiplying cyclically "from left to right", the third one obtained by multiplying cyclically from "right to left". In this case each triple involves nine multiplications. Of course also triples with higher differences can be studied, and triples with different differences can be mixed as well. Furthermore: Pairs and triples can be generalised to n-tuples. Also more advanced mathematics can be employed as the expressions are scalar products of vectors …

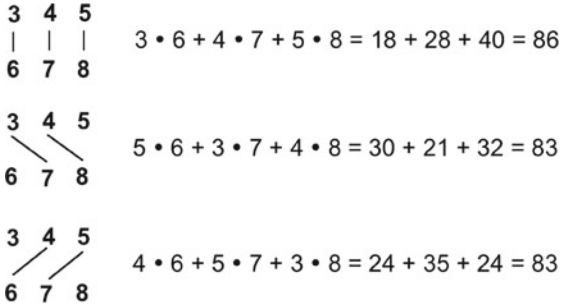


Fig. 8 Generalizing the problem

In grade 2 or 3 only a tiny section of this very substantial learning environment can be explored. However, this does not reduce its importance for mathematics education as will be shown in the final section.

5 Substantial Learning Environments in Teacher Education

> I never force a piece of wood into a salad bowl. It's a raw material, living and talking.
>
> P. Peeters, Belgian wood artist

Efforts to establish a systemic relationship between theory and practice must include teacher education as it is in this field that the foundations for being able to act as a reflective practitioner are laid. SLEs, if properly used, can play a fundamental role here, too. It is appropriate to discuss didactical and mathematical courses separately as their positions in teacher education are different.

5.1 Didactics Courses

The use of substantial learning environments is obvious for the didactic education of student teachers, that is for methods courses. By their very design SLEs offer unique possibilities for linking theoretical principles to concrete examples. If student teachers leave the university with the intimate knowledge of theory-based learning environments they have at their disposal a professional background that will help them immensely to act as reflective practitioners. As convincingly explained in Chap. 6 of Stigler and Hiebert (1999, p. 85 ff.) teaching is a cultural activity that can only be understood by becoming active in this culture. For this reason the best way for student teachers to capture the spirit of a substantial learning environment is to explore its epistemological structure, to reflect on it in terms of didactic principles, and to test their anticipations in the light of practical experiences. John Dewey gave a wonderful account of this "laboratory point of view" in his fundamental paper *The Relation of Theory and Practice in Education* first published almost 100 years ago (Dewey 1976).

In the last few years enormous progress has been made in applying the new technological possibilities of Multimedia to teacher education (cf. Lampert and Ball 1998). Here SLEs can be of great help in order to identify teaching episodes that are substantial, mathematically and didactically, theoretically and practically, and to establish a well-structured and manageable information system that reflects the contents, objectives and principles of teaching mathematics at the corresponding level.

5.2 Mathematics Courses

It is a simple matter of fact that around the world mathematical courses or even whole programmes often make only little or no sense for student teachers, for various reasons. Either the relevant subject matter is not covered at all, or the mathematical substance is stifled by a formalistic style of presentation or, even worse, there *is no substance*: mathematics is reduced to conceptual or procedural skeletons. Nevertheless, it would be wrong to conclude from meaningless courses that mathematical courses proper are of no use, in principle, for student teachers, and that the necessary mathematics should better be integrated into courses in mathematics education. On the contrary, a specific understanding of subject matter is of paramount importance for teachers as was convincingly explained, for example, by John Dewey in the paper mentioned above (Dewey 1976). Dewey's arguments are based on a genetic perspective. He saw scientific enquiry as a social process and knowledge as a result of it.

From this perspective Dewey's emphasis on teachers' subject matter knowledge must not be taken as an unconditional support for mathematical courses of any kind but for courses which meet specific criteria. Courses in the context of specialised mathematics are perhaps appropriate for prospective mathematicians in industry or in research. However, from the point of view of mathematics education, it is counterproductive to take such courses as a model for teacher education. To consider specialised mathematics as something absolute and as a yardstick for the mathematical training in any other professional context would be a fundamental mistake. It is a well established fact in the psychology of learning that knowledge cannot be acquired as a formal structure independently of the context in which it is to be used. Therefore, what is needed for teacher education is an idea of *mathematics in the educational context*, as formulated, for example, by Freudenthal:

> The idea of transposing academic mathematics (savoir savant) down to school mathematics (savoir enseigné) is wrong at its very outset, because the thinking behind this idea is directed top down and not bottom up. The mathematics to be learned at school by the big majority of our prospective citizens does not correspond at all to any theories of academic mathematics from which it could be watered down (didactically or not); at best it corresponds to the mathematics of scholars who lived centuries ago. The vast majority of our young people must be prepared to a technological know how (at various levels), not to the special knowledge of experts. The role of academic mathematics within this technological culture is much more modest than it has been claimed since a quarter of a century… (Freudenthal 1986, transl. E.Ch.W.)

To postulate a specific conception of mathematics in the educational context has implications for both contents and methods. Elementary topics which are closely related to the curriculum are far more important for teachers than advanced topics, and above all student teachers must experience mathematics as an *activity* (Freudenthal 1973; Wittmann 2001). It is only in this way that they can learn to deal with elementary mathematical structures in a productive way and to play their role as reflective practitioners also with respect to contents: SLE studies presuppose a flexible mastery of the content.

In order to make mathematical courses meaningful for teacher education they should be systematically related to SLEs. By their very definition SLEs are based on substantial mathematics *beyond* the school level. Therefore every SLE offers mathematical activities for student teachers on a higher level. However, disconnected pieces of mathematical islands attached to scattered SLEs do not serve the purpose. What is needed in teacher education are systematic and coherent courses of elementary mathematics which cover the mathematical background of a variety of SLEs. To develop such courses is a challenging problem for the next decade. Within the project Mathe 2000 a special series *Mathematics as a Process* has been started which is an attempt in this direction (cf., Müller et al. 2002).

Focusing the mathematical education of student teachers on substantial learning environments can also serve another purpose. At a time when education in general is in danger of being subordinated to economic purposes and to methods of mass production, when, as a consequence, mathematics at school is in danger of being reduced to a toolkit for applications, and teaching is in danger of preparing students just for passing tests, the following point is crucially important: Student teachers of all levels must experience the aesthetics of a genuine mathematical activity leading to the creation of structural wholes. The presence of SLEs in the mathematical studies can contribute to making student teachers aware that mathematics is not a homogeneous mass that can be cut into arbitrary pieces and forced into instructional schemes. Mathematical structures are living organisms, and learning processes must follow their inherent dynamics if learning mathematics is to make a deeper sense.

6 Conclusion

The systemic approach to the management of complexity on which this paper is based is more than just a scientific paradigm: basically, it is a way of life carried by an enlightened self-interest and directed towards sustainable development and co-existence. Therefore it is appropriate to conclude the paper with the systemic postulates for a future society formulated by Heinz von Foerster, the great master of systemic thinking (von Foerster 1984):

1. Education is neither a right nor a privilege: it is a necessity.
2. Education is learning to ask questions to which the answers are not known.
3. A is better off if B is better off.

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Chapter 10
The Alpha and Omega of Teacher Education: Organizing Mathematical Activities

> *In future not instruction and receptivity, but organisation and activity will be the special mark of the teaching/learning process.*
>
> *Johannes Kühnel (1869–1928)*

1 Introduction

The aim of this paper is to describe an introductory mathematics course for primary student teachers and to explain the philosophy behind it.

The paper is structured as follows: It starts with a general plea for placing the mathematical training of any category of students into their professional context. Then the context of primary education in Germany, with its strong emphasis on the principle of learning by discovery, is sketched. The third and main section of the paper presents the "O-script/A-script method", a special teaching/learning format for stimulating student teachers' mathematical activities along the principle of learning by discovery. In Sect. 4 special attention is given to the notion of proof in the context of primary teacher education. The paper concludes with some observations of how student teachers evaluate this approach.

2 Mathematics in Contexts

It is a most remarkable phenomenon that the teaching and learning of mathematics at the university level which was hardly a subject of public discussion in the past is now attracting world wide attention. The Discussion Document for the ICMI Study on this topic (ICMI 1997) lists five external reasons for this changing attitude:

1. the increase in the number of students who are attending tertiary institutions;
2. pedagogical and curriculum changes that have taken place at the pre-university level;

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3. the increasing differences between secondary and tertiary mathematics;
4. the rapid development of technology;
5. demands on universities to be accountable.

I would like to add an internal reason and to comment on it: the changing views about the nature of mathematics. The first three-quarters of this century witnessed a steady rise of formalism and structuralism culminating in Bourbaki's monolithic architecture of mathematics. However, by the end of the seventies this programme despite its success in some fields of mathematics turned out as a failure as a universal programme, as did similar structuralistic programmes in other areas, for example linguistics and architecture. At that time it was widely recognized that in no field of study could semantics be replaced by syntax. Postmodern philosophy rediscovered the meaningful context as an indispensable aspect of all human activity, including mathematical activity. As far as details of the changing views of mathematics are concerned I refer to Davis and Hersh (1981) and Ernest (1998).

As a consequence, we have to conceive of "mathematics" not solely as an academic field of study but as a broad societal phenomenon. Its diversity of uses and modes of expression is only in part reflected by the kind of specialized mathematics which we typically find in university departments. I suggest a use of capital letters to describe MATHEMATICS as mathematical work in the broad sense including mathematics in science, engineering, economics, industry, commerce, craft, art, education, daily life, and so forth, and including the customs and requirements specific to these contexts. Of course, specialized mathematics is a central part of MATHEMATICS. But mathematicians cannot and must not claim a monopoly for the whole. It is unjustified to assume that any piece of mathematics would form an absolute body of knowledge carrying its potential applications in itself. In his paper "The pernicious influence of mathematics on science" J. T. Schwartz used drastic words to warn mathematical specialists of applying mathematics to other fields without paying proper attention to the context (Schwartz 1986).

The consequences for the teaching and learning of mathematics at the university should be clear: In teaching mathematics to non-specialists the professional context of the addressees has to be taken fundamentally and systematically into account. The context of mathematical specialists is appropriate for the training of specialists, not for the training of non-specialists.

In the present paper the professional context to be considered is teaching mathematics at the primary level. There are mathematicians who look down on this task. In my view this is a fundamental mistake. The importance of primary mathematics within MATHEMATICS can hardly be overestimated. After all, it is at this level where the systematic encounter of children with mathematics begins and where the points for their whole mathematical education are set. I would like to refer here to the wisdom of the Tao-te-ching:

> Plan difficult things at the very beginning when they are still easy.
> Care for big things as long as they are still small.

Although many elements of the context of primary teacher education are specific the general approach adopted in this paper might be interesting for developing mathematical courses for other professional fields, too.

3 The Context of Teacher Education

Since the beginning of the 1980s the development of primary education in the State of North Rhine-Westphalia has exerted a great influence on the other German States.[1] The boundary conditions for primary mathematics education in North Rhine-Westphalia are special in two respects:

1. In the first phase[2] of their education at the university all primary student teachers have to study three subjects: German language, mathematics and a third subject (for example, environmental education, physical education, art, etc.). One of the three subjects has to be chosen as a major subject (45 credit hours out of the 120 credit hours of the whole 3-year programme). Two other (minor) subjects cover 25 credit hours.[3] As a consequence mathematics is compulsory for all primary student teachers. Roughly 90% of them choose mathematics as a minor subject (25 credit hours).
2. The syllabus for primary schools (grades 1 to 4) adopted in 1985 marked an important turning point in the history of public education in Germany. For the first time the principle of learning by discovery was explicitly prescribed as the basic principle of teaching and learning (Kultusminister des Landes Nordrhein-Westfalen 1985, Sect. 3):

> The tasks and objectives of mathematics teaching are best served by a conception in which learning mathematics is considered as a constructive and investigative process. Therefore teaching has to be organized such that children are offered as many opportunities as possible for self-reliant learning in all phases of the learning process:
>
> 1. starting from challenging situations; stimulating children to observe, to ask questions, to guess;
> 2. exposing a problem or a complex of problems for investigation; encouraging individual approaches; offering help for individual solutions;
> 3. relating new results to known facts in a diversity of ways; presenting results in a more and more concise way; assisting to memory storage; stimulating individual practice of skills;
> 4. talking about the value of new knowledge and about the process of acquiring it; suggesting the transfer to new, analogous situations.

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[1] With 17 million people Northrhine-Westfalia is the largest German State.

[2] The first phase (3 years) is followed by the second phase (2 years) which is spent at special institutions in close proximity of schools.

[3] 25 credit hours are for general education (pedagogy, psychology, ...).

> The task of the teacher is to find and to offer challenging situations, to provide children with substantial materials and productive ways of practising skills, and, above all, to build up and sustain a form of communication which serves the learning processes of all children.

This emphasis on mathematical processes instead of ready-made subject matter is visible in other parts of the syllabus, too. For example, the first section "Tasks and objectives" lists the following four "general objectives" of mathematics teaching: Mathematizing, Exploring, Reasoning and Communicating. Obviously, these objectives reflect basic components of doing mathematics at all levels. The fourth section of the syllabus describes in some detail why mathematical structures on the one hand and applications of mathematics on the other hand are two sides of one coin and how these two aspects can be interlocked in teaching. The explicit statement of this complementarity is also novel for German primary schools.

The development of this new syllabus was certainly very much influenced by similar developments in other European countries, in particular, the Netherlands. However, there has also been a strong trend towards active learning within German mathematics education. At the beginning of this century, Johannes Kühnel, one of the leading figures of progressive education in Germany, wrote his famous book "Neubau des Rechenunterrichts" ("Reconstructing the Teaching of Arithmetic") in which he described the "teaching/learning method of the future" as follows (Kühnel 1954, 70):

> The learner will no longer be expected to receive knowledge, but to acquire it. In future not instruction and receptivity, but organisation and activity will be the special mark of the teaching/learning process.

Since the late eighties considerable progress has been made in developing practical approaches and materials for this new conception of primary mathematics teaching including innovative textbooks (cf. Winter 1987; Wittmann and Müller 1994–1997, and Becker and Selter 1996). The project Mathe 2000 has played a leading role in this development. Of course the implementation of these materials depends crucially on the teachers' ability to abandon the deeply rooted instruction/receptivity model of teaching and learning in favour of the organisation/activity model. However, as experience shows, it is not enough just to describe new ways of teaching in general terms. The natural way to stimulate and to support the necessary change within the school system is to restructure teacher education according to the organisation/activity model. Only teachers with first hand experiences in mathematical activity can be expected to apply active methods in their own teaching as something natural and not as something imposed from outside. Therefore all efforts in pre-service and in-service teacher education have to be concentrated on reviving student teachers' and teachers' mathematical activity.

Interestingly, the new emphasis on student activity is not restricted to teacher education, it is a general phenomenon of the present discussion about teaching mathematics at the university level (cf. the section "Student Activity" in ICMI 1997). More and more mathematicians are taking special care of stimulating student activities. Bill Jacob's "Linear Functions and Matrix Theory" (Jacob 1995) is a good example.

4 The O-Script/A-Script Method

The traditional pattern of introductory mathematics courses at German universities is a combination of a 2 to 4 hours per week lecture ("Vorlesung") on the one hand and 2 hours of practice ("Übungen") which take place in groups of about 30 students on the other hand. I am well aware that expository teaching can be very stimulating and that work in groups based on good problems can arouse students' thinking and communication as well. Nevertheless I contend that *grosso modo* the lecture/practice pattern has a strong inherent tendency towards instuction and receptivity: Often the tasks and exercises offered to students for elaboration require mainly or even merely a reproduction of the conceptual and technical tools introduced in the lecture. So more or less students' individual work and work in groups tends to be subordinated to the lecture. Frequently, work in groups degenerates into a continuation of the lecture: The graduate student responsible for the group just presents the correct solutions of the tasks and exercises.

The lecture/practice format is particularly common in courses for large groups of students. In fact if you are confronted with numbers of students as large as 400 to 600, as we are in our primary teacher education programme, there is a strong pressure towards instuction/receptivity, and it is hard to think of alternatives.

However, the more I got involved in developmental research along the lines of learning by discovery the more I felt the contradiction between the teaching/learning model which I followed in my mathematical courses and the teaching/learning model which I recommended in my courses in mathematics education.

The O-script/A-script method has been developed as an attempt to mitigate this cognitive conflict. The basic idea, the **A**lpha and **O**mega, of this method is very simple: Just take Johannes Kühnel literally in teacher education and replace "instruction and receptivity" by "**O**rganisation and **A**ctivity", that is, use both the lecture and the group work for organizing student activities.

An essential ingredient of this new teaching/learning format is a clear distinction between the text written down by the lecturer on the blackboard or the overhead projector and the text elaborated by the individual student. As the lecturer's main task is to organize students' learning her or his text is called the "O-script". It is not a closed text, but it contains many fragments, leaves gaps, and often gives only hints. Therefore it is a torso to be worked on. As the elaborated text expresses the student's individual activity it is called the "personal A-script".

The regulations of our teacher education programme do not allow for making the A-script obligatory. However, the A-script can be used as an additional qualification by students who fail the final test. Experience shows that the majority of student teachers is willing to write an A-script. How to organize students' activity in a lecture? In trying to find an answer to this question I got inspired by two quotations:

> We should teach more along problems than along theories. A theory should be developed only to the extent that is necessary to frame a certain class of problems. (Giovanni Prodi)
> The main goal of all science is first to observe, then to explain phenomena. In mathematics the explanation is the proof. (David Gale)

Accordingly, I divided the course in two parts: The first part was devoted to introducing and clarifying a list of 50 carefully selected generic problems which should be elaborated in the A-scripts. The second systematic part should present a theoretical framework for these problems, however, based on students' experiences in writing the A-scripts. The second part did not differ from ordinary lectures. I think this format absolutely appropriate at this place of the learning process. Actually, I don't see a substitute for it.

The following areas which are closely related to the contents of the primary curriculum were covered in the course: (1) Place Value Systems, (2) Elementary Combinatorics, (3) Arithmetic Progressions, (4) Sequences, (5) Elementary Number Theory.

These areas are rich playgrounds for genuine mathematical activities. By using the opportunities offered in the course student teachers acquire not only the appropriate background knowledge which enables them to look at the primary curriculum from a higher level. They also acquire first-hand experiences in mathematizing, exploring, reasoning, and communicating.

The 10 problems selected for the area "Arithmetic Progressions" are as follows:

1. (From Butts 1973.) Try to decompose the set $\{1, 2, 3, ..., n\}$ of the first n natural numbers into two subsets such that the sum of the numbers in one subset is equal to the sum of the numbers in the other subset. For which n is this possible? For which n not?
2. Investigate the analogous problem for the set $\{2, 4, ..., 2n\}$ of the first n even numbers.
3. Investigate the analogous problem for the set $\{1, 3, ..., 2n - 1\}$ of the first n odd numbers.
4. Which numbers can be represented as sums of consecutive numbers?
5. Which numbers can be represented as sums of 2 (or 3, 4, ...) consecutive numbers?
6. In how many ways can 1000 be represented as a sum of consecutive numbers?
7. In how many ways can 1000 be represented as a sum of consecutive *odd* numbers?
8. From Monday to Friday 60 little lambs were born on a pasture: on Tuesday 3 more than on Monday, on Wednesday 3 more than on Tuesday, on Thursday 3 more than on Wednesday, and on Friday 3 more than on Thursday. How many lambs were born on each day?

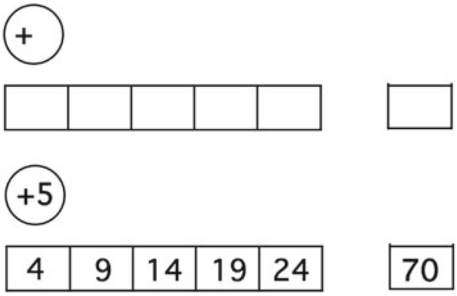

Fig. 1 Steinbring's problem

9. (From Steinbring 1997). In the scheme of Fig. 1 the number in the circle (the "addition number") and the number in the first box ("the starting number") can be chosen arbitrarily. The numbers in the other four boxes are calculated inductively according to the following rule (see the example in Fig. 1): The number in a box is the number in the preceding box plus the addition number. The numbers in all five boxes are added to give the final result (the "target"). How to choose the starting number and the addition number in order to get the target 50? How many solutions do exist? Which numbers can be obtained as targets?
   (In this problem and the next one natural numbers and the number 0 are admitted.)
10. Investigate the same problem for 6 boxes instead of 5.

The list of these 10 problems has been constructed by employing the "method of generating problems" (Wittmann 1971). So the use of heuristic strategies is ensured. Problem 8 is taken from a textbook for grade 4, problem 9 from a paper on the findings of a teaching experiment based on this problem. Therefore student teachers can see explicit connections with the primary curriculum.[4] As these connections are reinforced in the subsequent maths education course the maths courses become meaningful for student teachers within their professional context.

In the first part of the course each weekly lecture introduced 5 problems to the student teachers for investigation. The problems were explained in full detail and it was indicated how these problems could be attacked in different ways by using various "enactive", "iconic" and "symbolic" representations. The main heuristic strategies as described, for example, in Polya (1981), Mason (1982) or Schoenfeld (1985), were explained by referring to the problems of the course. However, no solutions were given.

The student teachers had less problems with developing ideas. The real challenge was how to formulate a coherent text. "What should an A-script look like?" was a frequent question. So parts of the lecture as well as of the group work had to address this difficulty. Referring to some examples I indicated in my lectures how the gaps of the O-script can be filled to get an A-script. In addition, student teachers were allowed to submit drafts of their A-scripts for critical reading, and could revise them according to the comments they received.

At the end of the first part of the course the students (at least the brave ones) had intensively worked on 50 selected problems. Even when they hadn't solved all problems properly, they had experienced a variety of mathematical phenomena. This was a good basis for the theoretical framework developed in the subsequent second part of the course.

For example, the problems on arithmetic progressions were theoretically framed by proofs of the sum formula and of the following remarkable theorem by J.J. Sylvester: The number of representations of a number n as a sum of consecutive numbers is equal to the number of odd divisors of n.

Both proofs were based on ideas that had been developed by students before.

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[4] After their own work on problem 9 student teachers were shown a video on a teaching experiment in which a group of 12 fourth graders had found all solutions within 30 minutes.

Interestingly, the extended work on problems in the first part paid off in the second part: the course "covered" the same mathematical content as courses in the ordinary format usually do.

5 Operative Proofs

As stated at the beginning, the basic tenet of the present paper is that the mathematical training of student teachers should reflect their professional context. This requirement is particularly critical when it comes to proofs.

It should be obvious that the notion of formal proof related to deductively structured theories is inappropriate or even counterproductive as a background for appreciating "Reasoning" as an objective of primary mathematics. That is not to say, however, that the notion of proof is irrelevant for primary mathematics. On the contrary. Fortunately, contemporary views of proof allow for an intellectually honest incorporation of proof into both primary teacher education and primary teaching. Studies in the history and philosophy of mathematics have destroyed the long held formalistic doctrine that the only rigorous form of proof is a formal proof. It has turned out that the notion of formal proof has its clear limitations, particularly from the point of view of the practising mathematician (cf., for example, Branford 1913, Hardy 1929, Thom 1973, Davis and Hersh 1981, Atiyah 1984, Long 1986 and Thurston 1994). In a letter submitted to the working group on proof at ICME 7, Québec 1992, Yuri I. Manin expressed his broader understanding of "proof as a journey" very nicely:

> Many working mathematicians feel that their occupation is discovery rather than invention. My mental eye sees something like a landscape; let me call it a "mathscape". I can place myself at various vantage points and change the scale of my vision; when I start looking into a new domain, I first try a bird's eye view, then strive to see more details with better clarity. I try to adjust my perception to guess at a grand design in the chaos of small details and afterwards plunge again into lovely tiny chaotic bits and pieces.
>
> Any written text is a description of a part of the mathscape, blurred by the combined imperfections of vision and expression. Every period has its own social conventions, and the aesthetics of the mathematical text belong to this domain. The building blocks of a modern paper (ever since Euclid) are basically axioms, definitions, theorems and proofs, plus whatever informal explanations the author can think of.
>
> Axioms, definitions and theorems are spots in a mathscape, local attractions and crossroads. Proofs are the roads themselves, the paths and the highways. Every itinerary has its own sightseeing qualities, which may be more important than the fact that it leads from A to B.
>
> With this metaphor, the perception of the basic goal of a proof, which is purportedly that of establishing "truth" is shifted. A proof becomes just one of many ways to increase the awareness of a mathscape.
>
> Any chain of argument is a one-dimensional path in a mathscape of infinite dimensions. Sometimes it leads to the discovery of its end-point, but as often as not we have already perceived this end-point, with all the surrounding terrain, and just did not know how to get there.
>
> We are lucky if our route leads us through a fertile land, and if we can lure other travellers to follow us.

In mathematics education this new view of proof has been reflected in many papers (cf., for example, de Villiers 1997). Based on Semadeni's and Kirsch's proposals of "pre-mathematical" or "pre-formal" proofs (Semadeni 1974; Kirsch 1979), the concept of "operative proof" has been developed (Wittmann 1997). An operative proof is a proof which is embedded in the exploration of a mathematical problem context and which is based on the effects of operations exerted thereby on meaningfully represented mathematical objects.

For this reason operative proofs explain phenomena which were observed before (cf. Gale's statement quoted above) and thus they contribute to understanding mathematics.

As also non-symbolic representations can be used operative proofs are particularly useful for the early grades and for primary teacher education. I would like to demonstrate this by giving two examples from my introductory course on arithmetic.

Example 1 (Infinity of primes) The formal proof of the infinity of prime numbers runs as follows: Let us assume that the set of prime numbers is finite: $p_1, p_2, \ldots, p_r$. The number $n = p_1 p_2 \ldots p_r + 1$ has a divisor p that is a prime number. Therefore n is divisible by one of the numbers $p_1, \ldots, p_r$. From $p|n$ and $p|p_1 p_2 \ldots p_r$ we conclude that p also divides the difference $n - p_1 p_2 \ldots p_r = 1$. However, $p|1$ is a contradiction of the fact that 1 is not divisible by a prime number. Therefore the assumption was wrong.

The following operative proof of the infinity of primes is based on the representation of natural numbers on the number line. One of the problems that the student teachers had to investigate was the determination of primes by means of the sieve of Eratosthenes. Therefore they knew from their own experience how the sieve works. Using this knowledge the infinity of primes can be proved just by explaining why the iterative sieve procedure does not stop: Assume that in finding primes we have arrived at a prime number p. Then p is encircled and all multiples of p are cancelled. The product

$$n = 2 \times 3 \times 7 \times 11 \times \ldots \times p$$

is a common multiple of all primes sieved out so far. So it was cancelled at *every previous step* of the procedure. As no cancellation process following the selection of a prime can hit adjacent numbers the successor of n has not been cancelled yet. Therefore after every step there are numbers left and the smallest of them is a new prime number.

Example 2 (Sylvester's theorem) In the first part of the course the student teachers worked with arithmetic progressions and investigated the representation of natural numbers as sums of consecutive numbers. Based on their experiences the following operative proof of Sylvester's theorem emerged in a natural way: Sums of consecutive numbers are represented as staircases. Depending on the parity of the number of stairs, each staircase can be transformed into a rectangular shape that represents a product. If the parity is odd, there is a middle stair and the upper part of the staircase can be cut off and added to the lower part (Fig. 2).

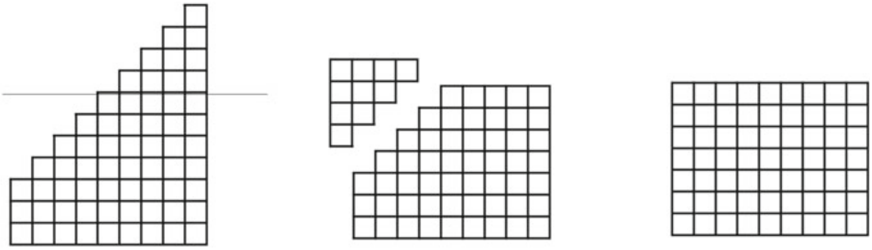

Fig. 2 Operative proof of Sylvester's theorem, case 1

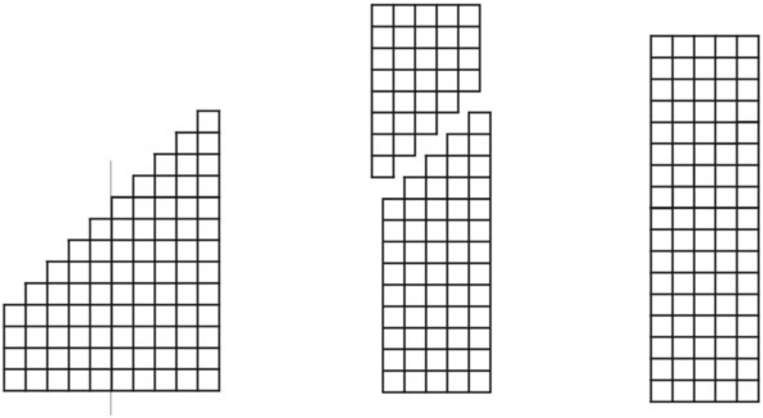

Fig. 3 Operative proof of Sylvester's theorem, case 2

If the parity of stairs is even then the staircase can be divided vertically in the middle and the two parts fit together to make a rectangular shape (Fig. 3).

A careful study of the effects of these two operations shows that in both cases an odd divisor of the represented number arises: either the number of stairs or the sum of the heights of the first and last stair (which must be odd for an even number of stairs). As a consequence any staircase representation of a number gives rise to an odd factor of n. But the converse is also true: A rectangle with an odd side can be transformed into a staircase of one of the two types depending on the relative size of the odd factor. A closer inspection reveals that this relationship between staircase representations and rectangular representations of n is bijective.

Again this operative proof explains phenomena which are well known from previous work on problems.

The advantage of operative proofs in the context of teacher education is obvious: These proofs are not separated from this context but closely related to it. In becoming acquainted with operative proofs student teachers learn to appreciate the use of informal means of representation for doing mathematics at early levels. Often, elements of such activities in teacher education can immediately be implanted into primary teaching. Consider, for example, the following exercise from a textbook for the second grade:

$$\begin{aligned} 1+2+3 &= \\ 2+3+4 &= \\ 3+4+5 &= \\ 4+5+6 &= \\ \dots \quad \dots & \end{aligned}$$

Looking at the results children discover the times 3-row. If the sums are represented by three columns of counters, the displacement of one counter to make a rectangle is obvious. This work with counters is a good and in my view also a necessary preparation for algebra where the same exercise can be resumed as follows:

$$(a-1)+a+(a+1)=3a.$$

6 Experiences with the Course

Feedback from student teachers collected by means of a questionnaire after the introductory course on elementary geometry showed that the "O-script/A-script" method was accepted by 75% of the population. The writing of the A-script was experienced as a very time-consuming, but effective exercise. In the same vein 70% affirmed that their understanding of the principle of learning by discovery had been improved.

However, only 59% of the students indicated that the course had had a more or less positive influence on their view of mathematics. 41% expressed their concerns about the openness of the first part. This result is not surprising as at school many students are programmed as receivers of knowledge. The adopted definitely mechanistic and formalistic attitude towards mathematics gives them a feeling of security and helps them "to survive". Feeling comfortable with mechanistic routines in the system of school and university (!) they do not want to be confronted with uncertainty.

The unfavourable influence of mathematical experiences from school is particularly apparent in student teachers' preconceptions of operative proofs. An instructive example was reported in Wittmann and Müller (1990). In a seminar student teachers studied figurate numbers.[5] In particular trapezoid numbers were introduced as a composition of square and triangular numbers (see Fig. 4).

In looking for patterns the students guessed that for all n the trapezoid number 'T_n and n leave the same remainder modulo 3. For this relationship an operative proof (at that time called "iconic proof") was given which was based on the corresponding pattern.

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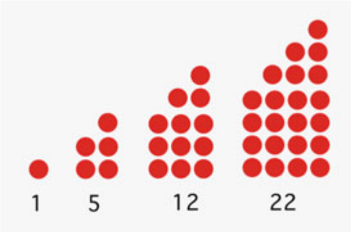
[5] In history figurate numbers played a fundamental role as a cradle of number theory. We are convinced that these numbers are also a wonderful context for stimulating mathematical activities in children. As a consequence figurate numbers play an important role in "mathe 2000".



$$T_n = n^2+(n-1)n/2 = (3n^2-n)/2.$$

Fig. 4 Trapezoid numbers

Right after this demonstration some students expressed their doubt on its validity. The teacher didn't intervene and quickly the whole group agreed that the demonstration could only claim the status of an illustration, not the status of a proof. The teacher then offered a formal proof and confronted it with the operative proof. The student teachers were invited to think about these two types of proof and to write down their opinions. The papers showed very clearly how the student teachers' appreciation of operative proofs was inhibited by the understanding of proof that they had acquired at school. For illustration I quote from some papers:

> The symbolic proof is to be preferred because it is more mathematical.
>
> The iconic proof is much more intuitive for me and explains much better what the problem is. For me the inferences drawn from patterns of dots are convincing and sufficient as a proof. Unfortunately we have not been made familiar with this type of proof at school. Only symbolic proofs have been taught.
>
> The iconic proof is very intuitive. One understands the connections from which the statement flows. I can't imagine how a counterexample could be found, because it does not matter how many 3-columns can be constructed. In my opinion it is nevertheless not a proof, but only a demonstration, which, however, holds for all n. At school I learned that only a symbolic proof is a proof.
>
> The symbolic proof is more mathematical. This proof is more demanding, as some formulae are involved which you have to know and to recall. The iconic proof can be followed step by step, and each step is immediately clear. However, I wonder if an iconic proof would be accepted in examinations.

Cognitive conflicts in accepting operative proofs as valid proofs have to be understood as natural symptoms of a metamorphosis lifting student teachers to higher professional levels. Experience shows that in retrospect student teachers consciously appreciate teacher education programmes which are embedded in the professional context. In a recent survey by the centre of teacher education at the University of Dortmund 2700 student teachers in North Rhine-Westphalia in their second phase of training were asked to evaluate the courses in mathematics and mathematics education they had received in the first phase of their training at the university (Zentrum für Lehrerbildung 1997). The results are very encouraging (Fig. 5). The evaluations of the programmes at the universities Paderborn and Dortmund which share the same philosophy were much higher than those of the six other universities in North Rhine-Westphalia which offer courses in primary teacher education.

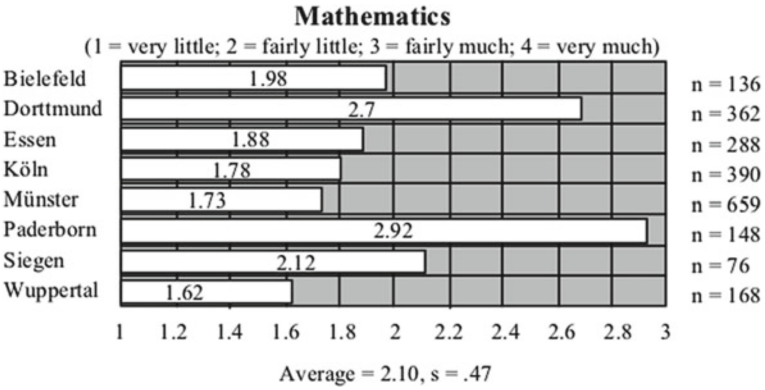


Fig. 5 Results of the empirical study

A team of 16 authors has just written a book "Arithmetic as a Process" (Müller et al. 2004) which is based on the O/A approach to teacher education described in this paper. This book is a truly mathematical book, but unlike other books it consciously puts mathematics in the context of teacher education—neither by sacrificing education to mathematics nor mathematics to education.

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Chapter 11
Operative Proofs in School Mathematics and Elementary Mathematics

Abstract This paper gives an account of the conceptual and practical approach to "operative proofs" that has been developed in the Mathe 2000 project. By means of some typical learning environments, this notion and its theoretical background are explained.

Keywords Operative proof · Learning environments · Practicing skills · Design science

In the past few decades, the notion of "proof" has been a prominent topic of research in mathematics education, both within the German-speaking world and at the international level. Within this topic, we can distinguish between three lines of research: the philosophical and epistemological aspects of proving, the elaboration of the various functions of proving, and the empirical investigation of students' routes to proving. Hanna and de Villiers (2012) provide an excellent overview of this research.

The present paper builds upon an independent line of research that has been evolving in German mathematics education in the context of elaborating on the operative principle and the genetic principle. This line of research is closely connected with curriculum development, and for this reason it is of particular interest for Mathe 2000, a project that is based on the following two basic assumptions:

1. A seamless learning process throughout a child's education is only possible if the teaching of mathematics from kindergarten through the end of high school is treated as a whole and if it reflects an authentic view of mathematics as the science of patterns (Wittmann 2006).
2. Mathematics education can best serve its purpose for developing mathematics teaching if it is conceived of as a "design science" (Simon 1970), that is, if the design, the empirical investigation and the implementation of the artificial objects of the design science mathematics education, namely substantial learning environments, are put at the very core of developmental research (Wittmann 1995, 2002).

In accordance with the first assumption, the project aims at introducing fundamental ideas of mathematics early and at developing them in a *genetic* way. Proving is one of these fundamental ideas. The investigation of this idea within the framework

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of ordinary teaching, namely by employing the usual means of representation and by connecting proof to the practice of skills, is a challenge on which we have focused.

It is on the second of these aspects, the practice of skills, that we have placed particular emphasis, as we see it as absolutely crucial to a successful and sustainable learning process. During the developmental research in Mathe 2000, the concept of "operative proof" has taken shape more and more. Papers by Werner Walsch and Heinrich Winter on proof, both related to curriculum development, have been important landmarks for us (see e.g. Walsch 1972; Winter 1984).

The structure of the present paper reflects the second basic assumption. The first section describes some learning environments which include "operative" proofs. These examples serve as illustrations for the second section, in which the notion of "operative proof" is explained, as well as for the last section, in which the theoretical underpinning of this notion will be described.

1 Some Learning Environments with Embedded Operative Proofs

The following four learning environments cover the spectrum from grades 1 to 6. At this level, the special features of operative proofs become particularly clear.

1.1 Even and Odd Numbers

Counters are a fundamental means of representing numbers in primary-school mathematics. Usually they are understood as "teaching aids" which have been specially invented for this purpose. However, their status is not primarily a didactic, but rather an epistemological one: in the time of Pythagoras there was a period in Greek arithmetic known as "*ψηφοι* arithmetic" which can be considered the cradle of arithmetic (Becker 1954, 34–41; Damerow and Lefèvre 1981).

In the Mathe 2000 curriculum, odd and even numbers are introduced in grade 1 in the ancient Greek fashion by means of special patterns of counters (Fig. 1).

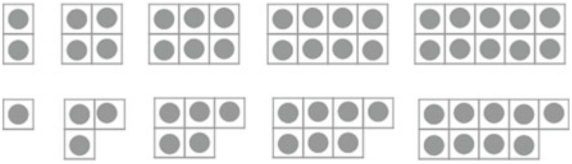

Fig. 1 Representation of even and odd numbers by dot arrays

These patterns are painted on cardboard and cut out so that children can perform operations with the pieces and form sums of numbers. The initial exercises help children become familiar with the material. The next exercise asks the children to find sums with an *even* result. This is a first invitation to look at the structure more carefully. The subsequent task is more direct, as children are asked to reflect on the results of the four packages of sums in Fig. 2: "*What do you notice? Can you explain it?*"

$4 + 6 =$	$5 + 1 =$	$2 + 1 =$	$1 + 8 =$
$6 + 8 =$	$7 + 3 =$	$4 + 3 =$	$3 + 6 =$
$8 + 4 =$	$9 + 5 =$	$6 + 5 =$	$5 + 4 =$
$10 + 2 =$	$5 + 7 =$	$8 + 7 =$	$7 + 2 =$
$12 + 8 =$	$9 + 9 =$	$10 + 9 =$	$9 + 0 =$

Fig. 2 Pretty packages with even and odd summands

At this early level, teachers are expected to refrain from pushing the children. All they should do is listen to children's spontaneous attempts to grasp the underlying patterns.

In grades 2 and 3, even and odd numbers are revisited using a wider range of numbers. This becomes necessary because there will inevitably be some children who will have to realize that 30, for example, is an even number although 3 is an odd number. Children are again given small packages of problems similar to those in Fig. 2 with larger numbers and asked the same questions.

At this level, the even/odd patterns are recognized more clearly and expressed in the children's own words more precisely. In the manual, teachers are advised to be content with children's spontaneous explanations and "warned" against demanding a "proof."

In grade 4, however, children are expected to have enough experience with even and odd numbers and to be ready to tackle the following task, which explicitly demands a proof:

Even numbers can be represented by double rows, odd numbers by double rows and a singleton. Use this representation to prove that:

(*a*) *The sum of two even numbers is always even.*
(*b*) *The sum of two odd numbers is always even.*
(*c*) *The sum of an even and an odd number is always odd.*

Children realize that no singletons occur when even patterns are combined and that in the case of two odd patterns, the two singletons form a pair, yielding another even result. Children also see that the singleton is preserved if an even and an odd pattern are combined and that in this case the result must be odd. The teacher's task is to take up the children's attempts and to assist the children in formulating coherent lines of argument.

The formal proof is addressed in higher grades, and in fact it expresses exactly the same relationships, albeit using a different language: the language of algebra. In general, operations with patterns of counters are an excellent preparation for algebraic calculations.

1.2 Multiplicative Arrow Strings

In grade 2 the multiplication of natural numbers is based on rectangular arrays of counters. The Hundred array (ten lines of ten dots, subdivided into four quadrants by the vertical and horizontal midlines) is a very convenient teaching aid. Children can easily represent and determine all products of the multiplication table. The subdivision of the field suggests the implicit use of the distributive law in calculating the results.

From the multitude of exercises, the following one is selected, in which children are offered strings of operators as in Fig. 3:

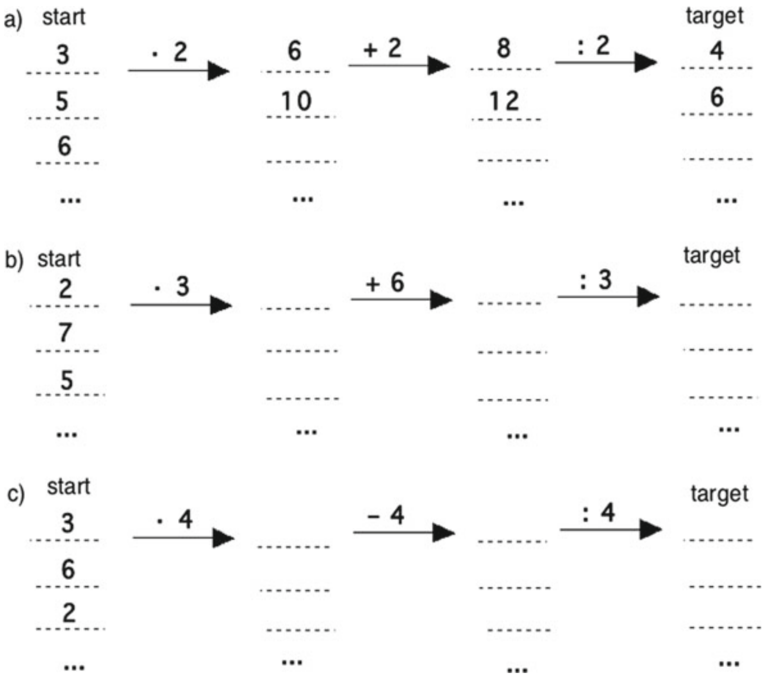

Fig. 3 Arrow strings

When reflecting on the results, students recognize that the target numbers differ from the starting numbers in a systematic way: in the first chain, the target number is always 1 more than the starting number; in the second chain, it is 2 more, etc.

An explanation of these number patterns can be given by referring to arrays of counters.

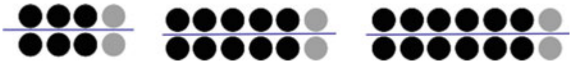

Fig. 4 An operative proof of the pattern underlying Fig. 3, a)

Figure 4 must be read as follows: We place 3 counters, double them, add two more counters, and finally divide by 2. We get one counter more than we had at the beginning. We start with 5 counters, double them, add two more counters and divide by 2. Again we get one counter more than we had before. We start with 6 counters, etc.

The repetition of the argument for several starting numbers is essential.

In grade 3, these operator chains are resumed with larger numbers; the operators ·2, +2 and :2 are replaced by the operators ·20, +20, :20, etc. The earlier explanations are repeated by referring again to arrays. This time, however, they are only given in shorthand notation (Fig. 5).

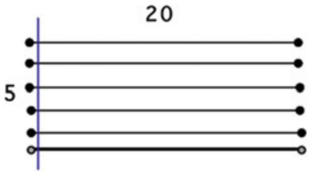

Fig. 5 An operative proof of the pattern with bigger numbers

The verbal description is as follows: “5 times 20 plus 20 is 6 times 20; 6 times 20 divided by 20 is 6, one more than 5” etc.

This argument is based on general relationships between numbers, not on special numbers. The approach provides efficient preparation for the transition to algebra long before variables are used.

1.3 Egyptian Fractions

This learning environment deals with a classic topic that can be found in some secondary-school textbooks, albeit without including a proof.

It is well known that the ancient Egyptians represented fractions smaller than 1 as sums of different unit fractions (with the numerator 1). To achieve this, they used a table for fractions of the type $2/2n + 1$. The mathematical question is whether any fraction smaller than 1 can be represented in this way. The answer is in the positive and the standard proof runs as follows: Let n/m be a reduced fraction, $n < m$. We choose the largest unit fraction $1/k$ smaller than n/m and subtract it from the given fraction:

$$n/m - 1/k = (n \cdot k - m)/m \cdot k.$$

The numerator $(n \cdot k - m)$ of the fraction on the right side must be smaller than n. Otherwise $1/k$ would not be the largest unit fraction smaller than n/m. Therefore, $(n \cdot k - m)/m$ is a fraction with a numerator smaller than n, and it is smaller than $1/k$. This procedure can be repeated. Step by step the numerators of the remaining fractions get smaller and smaller and, in a finite number of steps, one arrives at the numerator 1 and at a representation of n/m as a sum of different unit fractions.

There is another proof which rests on a repeated use of the formula

$$2/(2n + 1) = 1/(n + 1) + 1/(2n + 1) \cdot (n + 1).$$

The hard part, however, is showing that this algorithm terminates (see Fung 2005).

Both proofs go far beyond the secondary-school level, and again the question arises if it is possible to explain the existence of such a representation with elementary means. The following learning environment shows that it is possible.

First, students are provided with some historical background information. Then they are asked to find representations of reduced fractions of the type $2/3$, $2/5$, $2/7$, … as sums of different unit fractions. This investigation, which involves ample practice in adding and subtracting fractions, takes some time and leads to the following pattern:

$$2/3 = 1/2 + 1/6,\ 2/5 = 1/3 + 1/15,\ 2/7 = 1/4 + 1/28,\ 2/9 = 1/5 + 1/45,\ \ldots,$$

It is not important whether the students discover the underlying pattern themselves or whether the teacher provides some hints that incorporate the students' findings.

The explanation of this pattern is quite easy, as it rests on a very simple operative relationship: if the denominator of a fraction is increased, the fraction is decreased. If an arbitrary fraction of the type $2/2n + 1$ is given, say $2/31$, we increase the denominator by 1 and get a smaller fraction with an even denominator, $2/32$, and this fraction can be reduced to a unit fraction, $1/16$. Calculating the difference leads to

$$2/31 - 1/16 = (2 \cdot 16 - 31)/31 \cdot 16 = 1/31 \cdot 16 = 1/496,$$

which is a unit fraction. This procedure can be applied to *any* fraction of the type 2/2n+1. The numerator of the difference must always be 1, as it marks the difference between an odd number and the subsequent even number. Students should verify this fact by calculating quite a number of examples. In noting down the calculations on the blackboard, the table of the ancient Egyptians is re-established.

The next step is to look at reduced fractions of the type $3/n$, where n is not a multiple of 3. These are the fractions $3/4$, $3/5$, $3/7$, $3/8$, $3/10$, $3/11, \dots$

Again, the students' calculations can be ordered with the teacher's assistance. Perhaps some students will find out by themselves that the idea they have already applied to fractions with the numerator 2 can be adapted: Take any reduced fraction with the numerator 3, say $3/31$. Increase the denominator until you get a multiple of 3. The fraction $3/33$ is smaller than the given fraction and can be reduced to a unit fraction, $1/11$.

Calculating the difference leads to

$$3/31 - 1/11 = (3 \cdot 11 - 31)/31 \cdot 11 = (33 - 31)/31 \cdot 11 = 2/31 \cdot 11 = 2/341,$$

a fraction with the numerator 2. This difference can be treated as before:

$$2/341 = 1/171 + 1/341 \cdot 171 = 1/171 + 1/58\,311.$$

In this context, it becomes obvious that the numerator of the difference must be smaller than the numerator of the given fraction because it measures the distance of the denominator from the next largest multiple of the numerator. If the numerator of the difference is 1, the difference is a unit fraction and we are done. If it is 2, we can apply our earlier results for fractions with the numerator 2.

In the same way, fractions of the type $4/n$ can be reduced to fractions with the numerators 3, 2 or 1, and by mathematical induction we conclude that any reduced fraction smaller than 1 can be represented as a sum of different unit fractions.

Again, students should verify the procedure for quite a number of fractions. Example:

$$\begin{aligned} &5/11 - 5/15 = 5/11 - 1/3 = (15 - 11)/3 \cdot 11 = 4/33 \\ &4/33 - 4/36 = 4/33 - 1/9 = (36 - 33)/297 = 3/297 = 1/99 \end{aligned}$$

Therefore: $5/11 = 1/3 + 1/9 + 1/99$.

In order to double-check the calculations, the unit fractions should be added up, a useful exercise for adding fractions.

As in the previous learning environments, the notion of proof is not present at the beginning. It is only after quite a number of calculations that patterns are recognized and verified by checking the examples, and it is not until much later that these patterns are explained by looking at the effects of the operations. Practicing skills and proof are inseparably intertwined within a truly mathematical investigation.

1.4 *Fitting Polygons*

The following series of learning environments is based upon "fitting," a fundamental idea of elementary geometry (Freudenthal 1971, 422–423).

To develop the idea of "fitting" across the primary grades, the Mathe 2000 curriculum starts in grade 1 with the following activity (Fig. 6): paper squares of equal size are cut into two or four isosceles right triangles, and the parts are recombined to make other shapes.

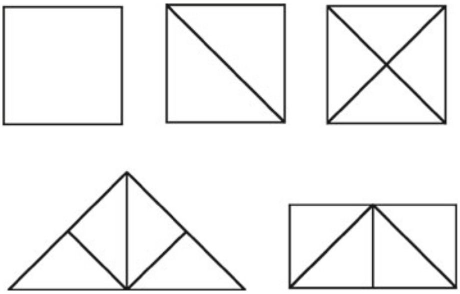

Fig. 6 Decomposing a square into isosceles triangles and rearranging the parts

At this level, the approach is essentially experimental. Students move the parts around and see if they fit. However, it is not merely experimentation that is at work here. For example, two right angles form a straight angle by the "definition" of a right angle. One of the shapes that can be obtained in this way is a special case of the Pythagorean theorem.

In grade 2 this activity is extended: paper squares are folded and cut so that four congruent right triangles are obtained (Fig. 7). One of the shapes that can be made by re-arranging these basic forms is well known as a foundation of the Pythagorean theorem.

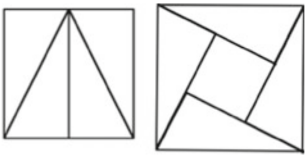

Fig. 7 Decomposing a square into four rectangles and rearranging the parts

Fitting regular polygons together is continued in grade 3 by means of a template for drawing squares, regular triangles, pentagons, hexagons and octagons with the same side length (Fig. 8). Children can explore, still experimentally, which shapes fit which way, a very creative exercise. They realize that there are only three regular tessellations, discover some semi-regular tessellations and quite a number of other tessellations.

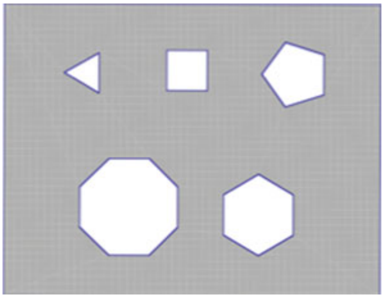

Fig. 8 Template for drawing regular polygons

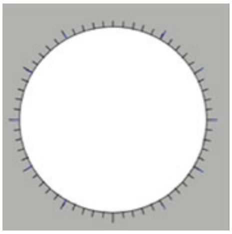

Fig. 9 “clock template”

In grade 4 children make regular polygons from cardboard by means of the “clock template” (Fig. 9) and build the five Platonic solids (Winter 1986). The name “clock template” is derived from the fact that a circle is divided into 60 equal parts. As 60 is divisible by 3, 4, 5 and 6, the clock template allows for a convenient construction of squares, regular triangles, pentagons and hexagons. For example, to draw a regular polygon one must divide the circumference into five equal parts of 12 “minutes” and connect the points. When clock templates of different sizes are used, polygons of different sizes are obtained. The shapes are copied onto cardboard. The segments of the circle containing the sides of the polygons can be folded down and used for pasting the polygons together. In this way, stable models of all five Platonic solids can be made. Interestingly, the proof of the existence of at most five Platonic solids at the end of book 13 of Euclid’s *Elements of Mathematics* is fully in line with the children’s experimental findings.

In grade 5 the concept and the measure of an angle is based on cutting and fitting polygons, which once again mirrors the historical development of mathematics (Becker 1954, 27). At this level, the purely experimental approach gives way to a conceptual approach: by referring to the measure of angles and the lengths of segments, students can explain why certain combinations of polygons must fit. Polygons cut from paper and drawings of polygons now enjoy a new status. They are no longer simple physical objects that allow for empirical experiments but rather representations of mathematical concepts that carry *theoretical* properties (see Sect. 3.2 below).

In the following grades, cutting polygons into parts and re-combining these parts is the customary approach to arriving at formulas for area and decomposition proofs of the Pythagorean theorem.

2 The Concept of Operative Proof

In the preface of Shafarevich 2005 there is an interesting statement concerning the limitations of formal definitions:

> . . . the meaning of a mathematical notion is by no means confined to its formal definition; in fact, it may be rather better expressed by a (generally fairly small) number of basic examples, which serve the mathematicians as the motivation and the substantive definition, and at the same time as the real meaning of the notion. Perhaps the same kind of difficulty arises if we attempt to characterize in terms of general properties any phenomenon that has any degree of individuality.

So it is for good reason that the present paper starts with typical examples of operative proofs. Referring to these examples, this notion can now be described as follows:

Operative proofs

- *arise from the exploration of a mathematical problem in the context of practicing skills and explaining patterns,*
- *are based on operations with "quasi-real" mathematical objects,*
- *use means of representation with which students are familiar at a given level*
- *are communicable in a simple problem-oriented language with little symbolism.*

Strictly speaking, the term "operative proof" is not entirely correct, as it is not the proof that is "operative" but rather the whole mathematical setting. However, for the sake of brevity the term seems acceptable. Operative proofs have received growing attention since Zbigniew Semadeni's seminal papers on "pre-mathematics" (Semadeni 1974; Semadeni 1984). His ideas were elaborated on in Germany by Kirsch (1979), Heinrich Winter (1985) and others and in Japan by Mikio Miyazaki (1997). These authors called proofs of this kind "pre-formal proofs" or "explanations by actions on manipulable things." These descriptions indicate that the authors had some concerns about the status of such proofs while, at the same time, they also had an unquestioned respect for formal proofs. However, research in the philosophy of mathematics and a re-thinking of the role of proofs in the mathematics community has changed the situation considerably cf. the overview given in Hanna 2000.

The first example in Sect. 1.1 shows that operative proofs are the most elementary form of proof associated with the first attempts to shape the discipline known as "mathesis." Operative proofs refer not to symbolic descriptions of mathematical objects within a systematic-deductive theory but rather directly to these objects via representations that allow for "concrete" operations. These operations are generally applicable independently of the particular objects to which they are applied. So it is not from particular cases that the generality of a pattern is derived but from *operations* with objects (see also Kautschitsch 1989, p. 184). This fact must be kept in mind in order to avoid erroneously rejecting "operative proofs" as non-rigorous proofs.

In higher mathematics the objects and the operations are much more complicated. Nevertheless, the operative character of proofs is still present in mathematics of all levels (see, for example, operative proofs of Sperner's Lemma in Struve and Wittmann 1984 and of the structure of the limit cycles of Bulgarian Solitaire in Wittmann 2006).

3 The Theoretical Background of Operative Proofs

The notion of operative proof is based on some theoretical positions from various disciplines. In this section, four positions will be described.

3.1 Mathematics as the Science of Patterns

As already mentioned at the beginning, the Mathe 2000 project has adopted the view of mathematics as the science of patterns, which has become a widely accepted view among mathematicians in the post-Bourbaki era (Sawyer 1995; Steen 1988; Devlin 1994). What matters in mathematics education, however, is not the science of ready-made and static patterns but rather the science of dynamic patterns which can be developed globally in the curriculum as well as explored, continued, re-shaped, and invented in the context of learning environments by the learners themselves. In other words, long-term and short-term mathematical processes related to patterns count much more than the finished products. The work of British, Scottish, Dutch and Japanese mathematics educators in the sixties and seventies as well as the pioneering work of Heinrich Winter, the "German Freudenthal," have all served as models (Fletcher 1965; Wheeler 1967; IOWO 1976; Becker and Shimada 1997; Winter 1984; 2015).

In order for students to understand mathematics, it is important that they become aware of mathematical patterns as early as possible. The ability to see something general in something particular is essential for appreciating and understanding mathematics at any level, particularly as far as the role of proof is concerned.

3.2 The Quasi-empirical Nature of Mathematics

Operative proofs depend on appropriate representations of mathematical objects. It was Imre Lakatos who first pointed out the fact that mathematical theories are always developed in close relationship with the construction of the objects to which they refer (Lakatos 1976). Graph theory emerges with the construction of graphs, group theory emerges with the construction of groups, theories of coding emerge with the construction of new codes, etc. In each theory the mathematical objects form a kind of "quasi-reality" which permits the researcher to conduct experiments similar to scientific experiments. In recent decades, the importance of this "quasi-empirical" perspective for mathematics education has gained more and more recognition.

At the school level, *informal* representations of mathematical objects are indispensable as they provide a "quasi-reality" that is easily accessible. Patterns become "visible" and managable when informal representations like counters, the number line, the place value chart, calculations with numbers and constructions of geometric figures are used.

Representations of mathematical objects, both informal and formal, form an interface between pure mathematics and managable applications. They can be seen as concretizations of abstract mathematical concepts on the one hand and as representations of real objects on the other hand. Compared with the abstract objects, these representations are more concrete than the mathematical objects which they represent, and compared with the real objects they model, they are more abstract.

The "quasi-reality" of mathematical objects forms a world of its own which Yuri Manin in a letter to ICME 7 aptly called a "mathscape." As the theoretical nature of mathematical objects is imposed on these representations, this mathscape is well suited to support the building of theories at whatever level by conveying meaning, stimulating ideas and providing data for checking mathematical arguments. Unlike Hilbert's fictitious mathematician who has cut all ontological links, the working mathematician and the learner act inside a "visible" mathscape. The following statement by D. Gale summarizes this position very neatly (Gale 1990, 4):

> The main goal of all science is first to observe and then to explain phenomena. In mathematics the explanation is the proof.

3.3 The Operative Principle

In Jean Piaget's epistemology, knowledge is seen as a construction that results from the interaction of the individual with the environment: the individual acts upon the environment, notices the effects of her or his actions, and fits them into growing and changing cognitive schemata. According to Piaget, mathematical knowledge is not derived from the objects themselves, but *from operations with objects* in the process of reflective abstraction ("abstraction réfléchissante," Beth and Piaget 1961, 217–223). Operations involve general patterns for the following reason: when it is intuitively clear that the operations applied to a particular object are applicable to all objects of a certain class to which the particular object belongs, the relationships which can be derived from these operations are recognized as generally valid.

Quite a number of German mathematics educators have contributed to applying Piaget's epistemology to mathematics education. Over the course of time, this has led to the following formulation of what is referred to as the "operative principle" (Wittmann 1996, 154–161):

> To understand mathematical objects means to explore how they are constructed and how they behave if they are subjected to operations (actions, constructions, transformations, …). Therefore students must be stimulated in a systematic way
>
> (1) to explore which operations can be performed and how they are linked to one another,
> (2) to find out which properties and relationships are imprinted into the objects through construction,
> (3) to observe which effects, properties, and relationships are brought about by the operations according to the guiding question "What happens with … if …?"

The relationship of this principle to operative proofs is obvious: operative proofs depend on the effects of operations applied to the objects in question. Because of the general nature of the operations, operative proofs are rigorous proofs with a clear foundation. At this level, the effects of the operations take over the role that axioms play at higher levels.

3.4 *Practicing Skills in a Productive Way*

When Mathe 2000 was founded 20 years ago, it was a conscious decision to pay particular attention to basic skills in order to escape the fate of many curriculum projects in the sixties and seventies which had failed because they neglected basic skills. Traditionally, "practice" is linked to the proverbial "drill and practice," which of course is not compatible with the objectives of mathematics teaching as we see them today. So a new approach to practice had to be developed which deliberately combines the practice of skills with higher objectives like mathematizing, exploring, reasoning and communicating. This type of practice has been called "productive practice" (Wittmann and Müller 1990/1992). The basic idea is quite simple: for practicing skills, appropriate mathematical patterns are used as contexts.

Learning environments designed accordingly always start with extended calculations, constructions or experiments. In this way a "quasi-reality" is created, allowing students to observe phenomena, discover patterns, formulate conjectures, and finally to explain, i.e. prove, patterns. The operations on which these operative proofs rest are introduced in this first phase in a natural way. Reference to this quasi-reality is made continuously while the environment is explored more and more deeply. In checking and verifying arguments, skills are practiced again.

The aspect of "practice" comes in a second time at a higher level. The ability to understand an argument in a proof depends on repetition as much as it does on the mastery of a skill. So it is very important that explanations are not only repeated several times within a given learning environment by referring to a series of different examples. It is equally important that coherent sequences of learning environments within the curriculum provide continued opportunities for repeating explanations. Heinz Steinbring's studies of the Mathe 2000 learning environments strongly confirm this fact (Steinbring 2005, Chap. 3). We cannot expect students to become familiar with operative proofs on the spot; students need continued opportunities for improving and refining their arguments. Developmental research in the Mathe 2000 project has shown that the addition table, the multiplication table, and the standard algorithms for addition, subtraction, multiplication and division are so rich in patterns that there is no need to introduce additional content for developing the higher objectives of mathematics teaching. It is crucial, however, to select representations of numbers that incorporate fundamental mathematical relationships and so to allow for operations upon which operative proofs can be built (Wittmann 1998). In arithmetic, counters provide the representation of choice. For example, rectangular arrays of counters allow the multiplication of natural numbers to be represented in a fashion that contains and supports the arithmetical laws. Section 1.2 provides some insight into the power of this representation.

4 Concluding Remarks

Operative proofs are not restricted to school mathematics but rather reach far into those parts of elementary mathematics that are accepted as the background of school mathematics and should form the subject matter of teacher education. The textbook "Arithmetic as a Process" (Müller et al. 2004), which was inspired by Mathe 2000, makes systematic use of informal representations and operative proofs within a process-led approach to the science of patterns. For example, in the chapter on number theory, all theorems up to Euler's generalization of Fermat's "little theorem" are explained by referring to "quasi-realities" represented by arrays of counters, the number line and arrays of numbers (Müller et al. 2004, 255–290).

In teacher education, the operative approach offers a dual advantage: this approach not only helps student teachers learn and understand mathematics better, but it also provides them with first-hand professional knowledge in dealing with means of representation and communication that are appropriate for the classroom. Mathematical courses designed accordingly provide an excellent basis for courses in mathematics education in which the underlying didactic principles can be made explicit by referring to student teachers' own mathematical experiences.

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Chapter 12
Collective Teaching Experiments: Organizing a Systemic Cooperation Between Reflective Researchers and Reflective Teachers in Mathematics Education

The success of any substantial innovation in mathematics teaching depends crucially on the ability and readiness of teachers to make sense of this innovation and to transform it effectively and creatively to their context. This refers not only to the design and the implementation of learning environments but also to their empirical foundation. Empirical studies conducted in the usual style are not the only option for supporting the design empirically. Another option consists of uncovering the empirical information that is inherent in mathematics by means of structure-genetic didactical analyses. In this chapter, a third option is proposed as particularly suited to bridge the gap between didactical theories and practice: collective teaching experiments.

The following five points indicate in a nutshell the line of argument of this paper.

- Mathematics education as a "systemic-evolutionary" design science
- Taking systemic complexity systematically into account: lessons from other disciplines
- Empowering teachers to deal with systemic complexity as reflective practitioners
- Collective teaching experiments: a joint venture of reflective researchers and reflective practitioners
- The role of mathematics in mathematics education.

1 Mathematics Education as a "Systemic-Evolutionary" Design Science

The proposal to consider mathematics education as a design science in Wittmann (1995) was stimulated by the intention to establish a sound methodological basis for a science of mathematics education that would guarantee a firm link between theory and practice and preserve the mathematically founded work achieved in curriculum development and teacher education by mathematics educators in the past (see Sect. 2). This proposal was based on the seminal book by Simon (1970) in which the design sciences were characterized as being concerned with the construction of artefacts that

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serve defined purposes. In the design science mathematics education these artefacts are substantial learning environments.

Therefore, the core of this discipline consists of the design, the empirical investigation, and the implementation of substantial learning environments both with respect to boundary conditions set by society and beyond these constraints. It is obvious that there is a basic difference between design sciences, such as mechanical engineering and computer science in which artefacts (cars, computers, etc.) are developed that function according to natural laws in a completely controlled way and can be easily applied by the users, and design sciences such as economics, medicine in which the artefacts (marketing strategies, therapies, etc.) cannot take account of all elements of the environment in which the artefacts are to be used as this environment is simply too complex and also fluid.

Following Malik (1986), these two classes of design science can be distinguished as "mechanistic-technomorph" and "systemic-evolutionary" design sciences. Obviously, mathematics education belongs to the latter class for which a sharp separation between researchers and developers who design artefacts and users who simply apply them is not appropriate. The consequences for mathematics education have been indicated already in Wittmann (1995) and further elaborated in more general terms in Wittmann (2001). In the following, the practical implications of this systemic principle are discussed.

2 Taking Systemic Complexity Systematically into Account: Lessons from Other Disciplines

In the comprehensive literature in which appropriate models for the cooperation between researchers and practitioners in systemic-evolutionary design sciences are developed, Donald Schön's research on the "reflective practitioner" stands out in depth and in scope (Schön 1983). Schön was mainly concerned with management, architecture, psychotherapy, town planning, and those parts of engineering in which social aspects matter. Later he extended his analyses also to education (Schön 1991). This stimulated other educators to expand on them (cf., for example Wieringa 2011).

Schön describes the traditional relationship between "professionals" and "clients" as follows (Schön 1983, p. 292):

> In the traditional professional-client contract, the professional acts as though he agreed to deliver his services to the client to the limits if his special competence … The client acts as though he agreed, in turn, to accept the professional's authority in his special field [and] to submit to the professional's ministrations.
>
> In some parts of some practices … practitioners can and do make use of the knowledge generated by university-based researchers. But even in these professions, … large zones of practice present problematic situations which do not lend themselves to applied science. What is more, there is a disturbing tendency for research and practice to follow divergent paths. Practitioners and researchers tend increasingly to live in different worlds, pursue different enterprises, and have little to say to one another.

Schön replaces the unproductive traditional roles of researchers and practitioners with a picture in which the responsibilities are to some extent shared. Researchers act as “reflective researchers” and practitioners as “reflective practitioners” (Schön 1983, p. 323):

> In the kinds of reflective research I have outlined, researchers and practitioners enter into modes of collaboration very different from the forms of exchange envisaged under the model of applied science. The practitioner does not function here as a mere user of the researcher’s product. He reveals to the reflective researchers the ways of thinking that he brings to his practice, and draws on reflective research as an aid to his own reflection-in-action. Moreover, the reflective researcher cannot maintain distance from, much less superiority to, the experiences of practice. …Reflective research requires a partnership of practitioner-researchers and researcher-practitioners.

However, Schön is far from denying researchers a special status: “Nevertheless, there are kinds of research which can be undertaken outside the immediate context of practice in order to enhance the practitioner’s capacity for reflection-in-action” (Schön 1983, p. 309).

Schön distinguishes four types of this “reflective research” (Schön 1983, p. 309ff.):

Frame analysis: This type of research deals with general attitudes that provide practitioners with general orientations for their work.

Repertoire-building research: The focus here is on practical solutions of exemplary problems (“cases”) that provide guidance not only in routine cases but also when it comes to dealing with similar new problems.

Research on fundamental methods of inquiry and overarching theories: This type is closely connected to both types mentioned above. It is directed to developing “springboards for making sense of new situations” for which no standard solution is available.

Research on the process of reflection-in-action: Here the emphasis is on stimulating and reinforcing practitioners to engage in reflective practice.

In recent years, the paradigm of applied science with its typical separation of responsibilities has been challenged also from another side. In his sociological studies of the ways technological tools (nuclear power stations, pesticides, vaccines, etc.) are developed, tested, and implemented and how these tools affect natural and social systems, the French philosopher Bruno Latour equally rejected the traditional separation between research and applications and introduced the concept of “collective experiment”:

> In this new constellation, the expert is more and more disappearing. … The expert has been responsible for the mediation between the producers of knowledge and the society concerned with values and ends. However, in the collective experiments in which we are intrinsically caught up, exactly this separation of different roles has disappeared. So the position of the expert has been undermined. [It has] been proposed that the extinct concept of “expert” be replaced by the comprehensive concept of “co-researcher.” (Latour 2001, p. 32, transl. E. Ch. W.)

Obviously the educational system is a “collective experiment, in which we are intrinsically caught up”. A separation between researchers who provide professional knowledge and teachers who simply use this knowledge is not appropriate.

Neither Schön's nor Latour's analyses provide practical solutions for mathematics education. However, they stimulate ideas for addressing the issue of managing complexity in this field.

3 Empowering Teachers to Cope with Systemic Complexity as Reflective Practitioners

In the first part of this section proposals are made how the collaboration between mathematics educators as reflective researchers and teachers as reflective practitioners can be filled with life. In the second part these proposals are examined in the light of the preceding section.

A good general orientation for this section is given by John Dewey's view on the role teachers can play as "investigators". This view bears witness to the systemic sensibility of this farsighted author:

> It seems to me that the contributions that might come from classroom teachers are a comparatively neglected field; or, to change the metaphor, an almost unworked mine. … There are undoubted obstacles in the way. It is often assumed, in effect if not in words, that classroom teachers have not themselves the training that will enable them to give effective intellectual cooperation. This objection proves too much, so much so that it is almost fatal to the idea of a workable scientific content in education. For these teachers are the ones in direct contact with pupils and hence the ones through whom the results of scientific findings finally reach students. They are the channels through which the consequences of educational theory come into the lives of those at school. I suspect that if these teachers are mainly channels of reception and transmission, the conclusions of science will be badly deflected and distorted before they get into the minds of pupils. I am inclined to believe that this state of affairs is a chief cause for the tendency, earlier alluded to, to convert scientific findings into recipes to be followed. The human desire to be an "authority" and to control the activities of others does not, alas, disappear when a man becomes a scientist. (Dewey 1929/1988, 23–24)

As stated in Sect. 1 the core of mathematics education as a design science consists of the design, the empirical investigation and the implementation of substantial learning environments with respect to boundary conditions set by society and beyond. So it has to be examined in which way teachers can be enabled and encouraged to act as reflective practitioners in these three areas.

In terms of design: In the author's view the most important service mathematics educators can render to teachers is to provide them with elaborated substantial learning environments together with the structure-genetic didactical analyses on which the design has been based. The language in which substantial learning environments are communicated is meaningful to teachers. So reflective practitioners have good starting points to transform what is offered to them into their context and to adapt, extend, cut, and improve it accordingly. In a recent paper Chun Ip Fung has demonstrated teachers' creative work in this area by means of a striking example and has shown that in this way a constructive dialogue between researchers and teachers can be established (see Sect. 3).

In terms of implementation: Individual learning environments and curricula cannot be implemented successfully without teachers' support. The implementation requires again teachers' creative powers in taking the local conditions into account and in adapting the proposed materials correspondingly. It is a triviality that teachers will engage more in the implementation of contents, objectives, or methods, the more these are meaningful to them. Reflective researchers have to keep this in mind.

In terms of empirical evidence: This is a particularly important issue. In the author's view teachers can best act as reflective investigators if empirical studies are attached to substantial learning environments and the results are communicable in a language that is understandable. Under these conditions teachers can cooperate in these studies and contribute to communicating the findings to practice.

However, empirical studies of the ordinary type are not the only way to get empirical evidence for the feasibility and the effectiveness of substantial learning environments. Another source are structure-genetic didactical analyses of the subject matter. Mathematics, well understood, provides not only the subject matter of teaching, but also methods of learning and teaching as it is itself the result of learning processes (see Sect. 4 in Wittmann 2018, with references to the fundamental paper by Dewey 1977). As these analyses imply empirical information on "staging" learning environments in the interaction with students, it is justified to call them empirical research "of the first kind," in distinction from ordinary empirical studies, the empirical research "of the second kind." Both structure-genetic didactical analyses and ordinary empirical studies are conducted either by researchers alone or determined by them. As teachers who collaborate with researchers in a research team are provided with additional information, have access to additional material, and enjoy support in various ways, they work under conditions that do not reflect the real practice. So for systemic reasons another type of empirical study seems promising: "collective teaching experiments." This empirical research "of the third kind" is obviously derived from Latour's "collective experiments." It is conducted by "freelancing" teachers in their daily practice, as will be discussed in some detail in the following section.

To conclude the present section, the above proposals for the interaction between reflective researchers and reflective teachers are examined against Schön's (1983) four types of "reflective research."

In terms of frame analysis: In order to provide teachers with an orientation beyond substantial learning environments, it is useful to summarize basic knowledge about mathematics, learning and teaching mathematics in didactical principles. One principle, for example, is "orientation on fundamental mathematical ideas." This principle is based on Alfred N. Whitehead's view on mathematical education (Whitehead 1929), Jean Piaget's epistemology (e.g. Piaget (1972), and Hans Freudenthal's work, in particular Freudenthal (1983). This principle can be communicated to teachers best by linking it to series of learning environments in which this principle is a leading one.

In terms of repertoire-building research: Elaborated substantial learning environments form a repertoire for teaching par excellence. They contain the essential information for teaching. The reflective teacher, however, will not stick to this repertoire but use it as a springboard for exploring other learning environments.

In terms of research on fundamental methods of inquiry and overarching theories: In close connection with the two types of research discussed before this type of research is directed to introducing teachers into methods of inquiry inherent in mathematics and into elementary mathematical theories of subject matter that are relevant for teaching.

In terms of research on the process of reflection-in-action: The proposals that have been made for the design, the empirical study, and implementation of a substantial learning environment are well suited to stimulating teachers to act as reflective practitioners.

It is obvious that both pre-service and in-service teacher education play a key role in educating reflective practitioners. Therefore the reflective mathematics educator is well advised to link his research to teacher education including mathematics education and at least elementary mathematics.

4 Collective Teaching Experiments: A Joint Venture of Reflective Teachers and Reflective Researchers

The idea to encourage teachers to become researchers of their own practice is not new at all. It is particularly manifest in the Japanese tradition of lesson studies (Stigler and Hiebert 1999). In lesson studies, a group of teachers collaborates over a period of time on the design, the empirical investigation, and the implementation of learning environments. The lessons are given by teachers in actual classrooms, observed, discussed and refined in several rounds until an acceptable result has been reached. A striking example is the recent Japanese research on elements of knot theory (Kawauchi and Yanagimoto 2012).

Collective teaching experiments are a modification of lesson studies in the following way: The reflective researchers offer research problems publicly and invite teachers to investigate them in their daily practice. There is only a loose connection with researchers who collect the feedback and turn it into the improvement of the design and the implementation.

The following example is to illustrate this proposal:

In the past decades, German math teaching at the primary level has undergone a development away from standard procedures towards flexible strategies that reflect the true nature of mathematics. The curriculum developed in the project Mathe 2000 is based on fundamental ideas of mathematics that can be developed over the grades. The arithmetical laws represent such a fundamental idea. The commutative and associative law of addition are implicitly introduced even at the kindergarten level and applied in a consequent and consistent manner at the primary and secondary level. The laws leave space for applying them in different ways, and it is important that teachers and children become aware of this freedom by being offered different strategies.

In adding two digit numbers there are essentially three basic strategies. None of them causes problems (see Fig. 1).

First plus tens, then plus ones		Tens plus tens, ones plus ones		Auxillary problem	
47 + 35 = 82	65 + 28 = 93	47 + 35 = 82	65 + 28 = 93	47 + 35 = 82	65 + 28 = 93
47 + 30 = 77	65 + 20 = 85	40 + 30 = 70	60 + 20 = 80	45 + 35 = 80	63 + 30 = 93
77 + 5 = 82	85 + 8 = 93	7 + 5 = 12	5 + 8 = 13		

Fig. 1 Basic strategies for addition problems

All three strategies can be transferred to subtraction. However, the second strategy causes a problem when the ones in the subtrahend exceed the ones in the minuend (see Fig. 2).

First minus tens, then minus ones		Tens minus tens, ones minus ones		Auxillary problem	
87 – 35 = 52	65 – 28 = 37	87 – 35 = 52	65 – 28 = 37	87 – 35 = 52	65 – 28 = 37
87 – 30 = 57	65 – 20 = 45	80 – 30 = 50	60 – 20 = 40	85 – 35 = 50	65 – 25 = 40
57 – 5 = 52	45 – 8 = 37	7 – 5 = 2	5 – 8 = –3		40 – 3 = 37

Fig. 2 Basic strategies for subtraction problems

Experience shows that many children transfer the strategy "tens plus tens, ones plus ones" blindly to the strategy "tens minus tens, ones minus ones" and arrive at wrong results. For the problem $65 - 28$, for example, they calculate $60 - 20 = 40$, $8 - 5 = 3$ and get 43. So teachers, supported by textbook authors, reject and avoid this strategy either by prescribing the first subtraction strategy or by modifying the critical one as follows: $50 - 20 = 30$, $15 - 8 = 7$, so $65 - 28 = 37$.

For us such didactic compromises are no option. We believe that it is better not to avoid the critical strategy also for its long-term importance. As early as 1977, the Dutch computer scientist Sytze van der Meulen, after his talk in our colloquium at our Institute for Development and Research in Mathematics Education in Dortmund, left a message in our guestbook that has since been a continuous reminder to us:

> When a boy answers the question "how much is $7 - 4$" with 3, he is not a genius when his age is 7. When this boy answers the question how much is "$4 - 7$" with "there are three missing" he shows some intelligence, but still is not a genius at the age of 7. The tragedy of our school-education is that this boy at the age of 11 may have difficulties with the concept of negative numbers. The tragedy of his teacher is that he missed 4 years of the boy's development!

Over the years we have taken several steps to overcome teachers' scruples concerning the subtraction strategy "tens minus tens, ones minus ones," and we have stimulated teaching experiments on a small scale. Since 1995 we have been using any opportunity to explain this strategy to teachers and to ask them to try it out with their students.

We recommend to explain $5 - 8 = -3$ as follows:

> We have 5 and have to take away 8. First we take away 5, and then we have to take away 3 more. In order not to forget this, we note it down as "−3". Finally we take away 3 by breaking up one ten into 10 ones, and remove 3 of them.

By means of bars of ten and counters, this procedure can be well demonstrated step by step.

We also tell teachers that this strategy has an important advantage: The calculations are easier in comparison with the first strategy, so this strategy seems particularly suited for weaker students despite the first impression that it might not be appropriate for them.

One teacher did a small study and communicated it to us: After she had taught subtraction in the hundreds space in the traditional way without the critical strategy, she administered a test to her class. Then she introduced this strategy and repeated the test. It turned out that the results were no better and no worse. As she had a class with many weak students who had difficulties with this strategy, her recommendation was to avoid this strategy. Nevertheless, we continued our "propaganda" for this strategy and improved our proposal how to explain it to students.

One year later an "unforced" e-mail arrived from this teacher that read as follows:

> Last year my students did have difficulties with the strategy "tens minus tens, ones minus ones" because of the negative numbers. Now I would like to report on my latest experiences with this strategy in grade 3. In the introductory lesson, I wrote down the problem $629 - 263$ without any repetition of the calculations from the year before and without any explanations from my side. Apart from very few exceptions the children calculated $20 - 60 = -40$. For them it was obvious: "The result is −40, exactly as last year with the ones." I would emphasize that my class is not a superclass, and that I have very many weak students. For them, calculations with negative numbers do not cause any problem. I am strongly in favour of introducing this strategy already in Grade 2.

As many teachers have still reservations against this strategy, we have refined our explanation. We recommend now to let students distinguish between the cases when there are enough ones in the minuend and those where more ones have to be taken away than are available. We recommend to propose packages of subtraction problems to students and asking them to mark those where a minus sign appears in ones calculation with an asterisk before they perform the actual calculations. The latest improvement in teaching this strategy is to explicate $5 - 8 = -3$ in more detail: $5 - 5 - 3 = 0 - 3 = -3$. We do not have enough feedback from teachers as this moment; however, we are confident that this step will increase the acceptance of this strategy.

These experiences and similar ones with other issues have led us to a far-reaching conclusion: Our main publication, a handbook for teaching arithmetic at the primary level (Wittmann and Müller 1990/1992) will be rewritten soon with the explicit invitation to teachers to conduct collective teaching experiments. All learning environments collected in this new book will belong to the standard curriculum. They will be accompanied not only with the general recommendation to read them critically and to test them in their classroom but also to participate in conducting collective teaching experiments in cooperation with other teachers. We will also create a platform for an exchange about the experiences with these experiments.

Issues that are of particular interest for us are operative proofs, the use of our course on mental arithmetic, and the use of new digital means of representation.

5 Closing Remarks: The Role of Mathematics in Mathematics Education

It is important to realize that the research and development program that has been described in this paper heavily depends on resources that are offered by "well-understood" mathematics. "Well-understood" means that mathematics is seen as a social organism that has developed in history and it still developing with strong relations to many areas of human life, and that also the mathematical knowledge of the individual is seen as an organism in its genesis from tiny seeds to a more or less extensive body. Doing mathematics is learning mathematics and learning mathematics should also be firmly linked to doing mathematics. Therefore, the interaction between teachers and students and between teachers and researchers can greatly profit from relying on the adaptability of elementary mathematical structures with respect to students' individual cognitive levels and on the processes inherent in vital mathematics.

When once asked what his motives as a mathematician were for engaging in mathematics education Hans Freudenthal replied: "I want to understand better what mathematics is about." The reverse also holds: mathematics educators who want to understand better what mathematics education should be about are well advised to study elementary mathematical structures thoroughly. It is highly rewarding to "unfreeze" the educational material that is "deep-frozen" in polished presentations of mathematics, as they are common in higher mathematics. After all "well-understood" mathematics is the best common reference for all involved in teaching and learning mathematics: researchers, teachers and students. "Theories of mathematics education" like those collected in Sriraman and English (2010) are far from being suited for establishing a systemic cooperation between reflective researchers and reflective teachers.

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Chapter 13
Structure-Genetic Didactical Analyses—Empirical Research "of the First Kind"

Abstract In mathematics education, theories of teaching and learning based on disciplines different from mathematics ("imported" theories) are widely dominating the field. This imbalance greatly reduces the impact of mathematics education both on teacher education and on the teaching practice. In order to return to a balanced situation it is necessary to pay more attention to theories which are based on mathematics. As an example of such a "homegrown" theory, the paper presents the structure-genetic didactical analysis, the research method of mathematics education conceived of as a "design science".

Keywords Design science · Substantial learning environments · Didactical analysis · Empirical research

AMS (2000) Subject Classification: Primary 97C · Secondary 97D

A comparison of the papers published in journals and proceedings in the 1970s and early 1980s (see, for example, the pace-setting paper Krygowska 1972) with the papers in the new millennium shows that over the past two decades the coordinate system of mathematics education has shifted massively *away from*

- the subject matter mathematics,
- the teaching practice and
- the critical examination of educational foundations concerning the subject,

towards

- qualitative and quantitative empirical studies of learning and teaching processes,
- the development and application of tests and
- theories of learning mathematics based on ideas imported from other disciplines.

This shift consciously or unconsciously involved a break from the tradition of mathematics education. Nevertheless, this tradition is still alive. In recent decades a branch of mathematics education has developed that explicitly builds on the tradition of "subject matter didactics" as it has been common in the past in many countries. This "mathematics education *emerging from the subject*", as it has been called, continues to carry the teaching of mathematics and teacher education and has created

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a scientific basis of its own. The internationally known project Mathe 2000 may serve as an example (Wittmann 2012). “Mathematics education *emerging from the subject*” constitutes by no means “didactics from the armchair” for which its predecessor had been criticized. On the contrary, it is supported empirically in its own way. Its specific feature is that it rests on theories of teaching and learning that are *implicit* in the subject mathematics itself. This will be shown in this paper, which is structured as follows: in the first three sections three central themes of the curriculum will be considered both from the position of the present mainstream in mathematics education and from the position of mathematics education *emerging from the subject*. In the fourth section the research method of the latter, the *structure-genetic didactical analysis*, will be characterized and it will be indicated what can be achieved by this method.

1 Introduction of the Multiplication Table in Grade 2

In the curricula of many countries multiplication is introduced as “repeated addition” and the multiplication table is accordingly learned row by row. The last decade has seen a vivid discussion in the Anglo-Saxon countries about what multiplication is about. The empirical analysis of (Park and Núñes 2001) fits into this context. The authors compared two hypotheses of concept formation for multiplication: multiplication as “repeated addition” and multiplication as a “schema of correspondences”. What the latter means, however, remains unclear in that paper. It is likely that the authors allude to the interpretation of multiplication as a linear function: for a fixed multiplier c we have a mapping that assigns the product $x \cdot c$ $(= c \cdot x)$ to any number x. As a result of their research the authors arrive at the conclusion that “repeated addition” should not be used for defining multiplication, but only for calculating the results.

From the perspective of mathematics education *emerging from the subject* multiplication in grade 2 can be approached in the following way: multiplication is defined as “abridged” addition, as it is common in mathematics. For calculating the results it is natural to refer to the laws of multiplication: among the multiples

$$1 \cdot m,\ 2 \cdot m,\ 3 \cdot m,\ 4 \cdot m,\ 5 \cdot m,\ 6 \cdot m,\ 7 \cdot m,\ 8 \cdot m,\ 9 \cdot m \ \text{ and } \ 10 \cdot m$$

there are four multiples that are trivial or easy to calculate:

$$\mathbf{1} \cdot m,\ \mathbf{2} \cdot m \text{ (double of } 1 \cdot m),\ \mathbf{10} \cdot m \ \textit{and} \ \mathbf{5} \cdot m \text{ (half of } 10 \cdot m).$$

Other multiples from $3 \cdot 7$ to $9 \cdot 7$ can be derived from easy ones by means of the distributive law:

$$\begin{aligned}
3 \cdot m &= \mathbf{2} \cdot m + \mathbf{1} \cdot m,\\
4 \cdot m &= \mathbf{2} \cdot m + \mathbf{2} \cdot m \ (or\ \mathbf{5} \cdot m - \mathbf{1} \cdot m),\\
6 \cdot m &= \mathbf{5} \cdot m + \mathbf{1} \cdot m,\\
7 \cdot m &= \mathbf{5} \cdot m + \mathbf{2} \cdot m,\\
8 \cdot m &= \mathbf{10} \cdot m - \mathbf{2} \cdot m,\\
9 \cdot m &= \mathbf{10} \cdot m - \mathbf{1} \cdot m.
\end{aligned}$$

This approach has been elaborated by Arnold Fricke in his "operative didactics" and is widespread in German primary schools (Fricke 1968). In the early eighties Heinrich Winter went one step further: In line with his general postulate to look at arithmetic from the point of view of algebra he suggested to use rectangular arrays of dots for representing multiplication (Winter 1984). This proposal is also found in Courant and Robbins (1996, p. 3), a classic among mathematical textbooks, and in Freudenthal (1983, pp. 109–110). In (Penrose 1994, pp. 51–53) it is even stated that rectangular arrays of dots are the most efficient means to explain what multiplication is about.

The preference of eminent mathematicians for these arrays underlines the fact that this representation of multiplication is not just a visual aid which has been invented for the purpose of teaching, but that is fundamentally interwoven in the epistemological structure of mathematics. The great advantage of this representation is that the commutative law, the associative law and the distributive law can be derived in an operative way and used in teaching (see, for example, Wittmann and Müller 2017, pp. 201–211). This is not possible with other representations of multiplication.

Later in the curriculum arrays of dots pass into the representation of a product as the area of a rectangle and this representation reaches up to the integral. It is a fundamental idea of algebra and calculus.

Comparison: What multiplication is about and how it should be introduced in the classroom, cannot be decided by means of empirical methods imported from psychology, but should be based on a sound mathematical and epistemological analysis. This, however, is not to say that empirical investigations of learning processes are superfluous (see Sect. 3).

2 Designing a Substantial Learning Environment for Practicing Long Addition

While the first example deals with the didactical foundation of some topic the second example leads to the very core of teaching. The natural way to help learners to master some piece of knowledge or some skill is to offer them substantial learning environments that stimulate mathematical activities. Here the practice of skills plays a crucial role. Heinrich Winter introduced the concept of "productive practice" which means a type of practice in which contents and general objectives of mathematics teaching (mathematizing, exploring, reasoning and communicating) are combined (Winter 1984).

In order to design a substantial learning environment for practicing long addition in our project Mathe 2000 we had to browse elementary mathematics for patterns that involve long addition. We had to check whether children’s knowledge in grade 3 is sufficient for understanding and solving the intended tasks, for exploring, discovering and describing patterns and for explaining them by using familiar means with some support of the teacher.

Our analyses led us to the following learning environment that is based on the famous rule “casting out nines” (Wittmann and Müller 2012, pp. 85).

The guiding problem posed to students is as follows:

> Form two three-digit numbers with the six digit cards 2, 3, 4, 5, 6, and 7 and add these two numbers.
>
> (a) Find different results.
> (b) Try to reach results as near as possible to 600, 700, 800, 900, 1000, 1100, 1200 and 1300.
> (c) Try to find results between 900 and 1000.

The subtasks (b) and (c) are intended as hints for discovering the underlying pattern.

Guy Brousseau’s theory of didactical situations provides a natural framework for the teacher in putting a learning environment into practice (Brousseau 1997).

Here this theory can be applied as follows: In the first situation the problem is introduced to students, best by means of examples.

In the second situation students work on their own, individually or in groups. The teacher serves as an advisor.

In the third situation the results are collected and compared. The teacher is free to add some more examples, and to give hints that stimulate students to discover the underlying pattern. Subtask (b) is particularly helpful as the optimal results 603, 702, 801, 900, 999, 1008, 1098, 1107, 1197, 1206, 1296, 1305 reveal a striking pattern: The total of the digits of these numbers is 9, 18 or 27.

The results in subtask (c) support these findings. Possible results are 900, 909, 918, 927, 936, 963, 972, 981, 990, 999.

A check with other examples will confirm this pattern. Of course some students will offer calculations with results that seem to violate this pattern. However, checks will reveal mistakes in the calculations.

In this way the conjecture is formed that for this problem only results are possible for which the total of the digits is a multiple of 9.

Situation 4 in Brousseau’s classification requires the explanation of this pattern. The place value chart with which students in grade 3 are familiar, serves this purpose perfectly (Wittmann and Müller 2013, 120–121): Some examples are represented by means of counters on the place value chart. It is interesting to note that in this context the total of the digits of a number has a very concrete meaning: It denotes the number of counters that are necessary for representing the number on the place value chart.

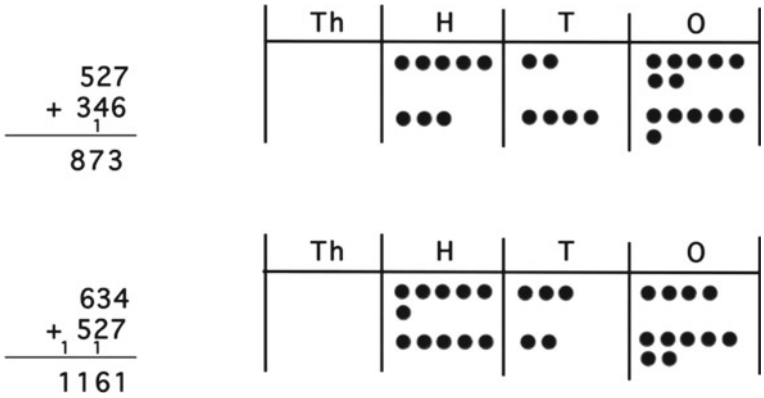


Fig. 1 Operative Proof of the rule "Casting out nines"

Figure 1 shows two examples:

In the first example 5 + 2 + 7 = 14 counters are needed to represent the first number 527 on the place value chart, and 3 + 4 + 6 = 13 counters are needed to represent the second number 346. So 14 + 13 = 27 counters are needed to represent the sum 527 + 346. To execute this addition on the place value chart means to push the counters in all columns together, and to replace 10 counters in the Ones column by 1 counter in the Tens column. Therefore 9 counters less 27 are needed to represent the result 873, namely 18 counters.

In the second example again 27 counters are needed to represent the sum. We have a carry from the Ones to the Tens column and a second carry from the Hundreds to the Thousands column. According to the two carries the total of digits of the result 1161 is $27 - 2 \cdot 9 = 9$.

As in all examples 27 counters are needed to represent the sum the total of the digits of possible results must be 27, 18 or 9 depending on the number of carries.

The fifth and final didactical situation is "institutionalization". Here the teacher's task is to summarize in a concise way what has been discovered. This might include the information that the operation of "casting out nines" is independent of the special numbers used here: For any sum of two or more numbers the sum of the totals of the digits of the numbers differs from the total of the digits of the result by a multiple of 9. The reason is that any carry involves a "loss" of 9 counters.

The teacher should also have in mind that this operative proof of the rule "casting out nines" is not an impasse, but that it can be continued later in the curriculum for deriving divisibility rules (Winter 1983).

Comparison: In this example the "home-grown" approach is unrivaled. It is obvious that theories of mathematics education imported from elsewhere, as well as empirical methods, are blunt when it comes to designing substantial learning environments. Only a thorough knowledge of mathematical structures and processes connected with curricular expertise will lead to solutions, and this knowledge is also essential for the teacher in doing her or his job.

3 Nets of a Cube

Nets of the cube are a standard topic of mathematics teaching at the secondary level. In this section two approaches to this topic are compared.

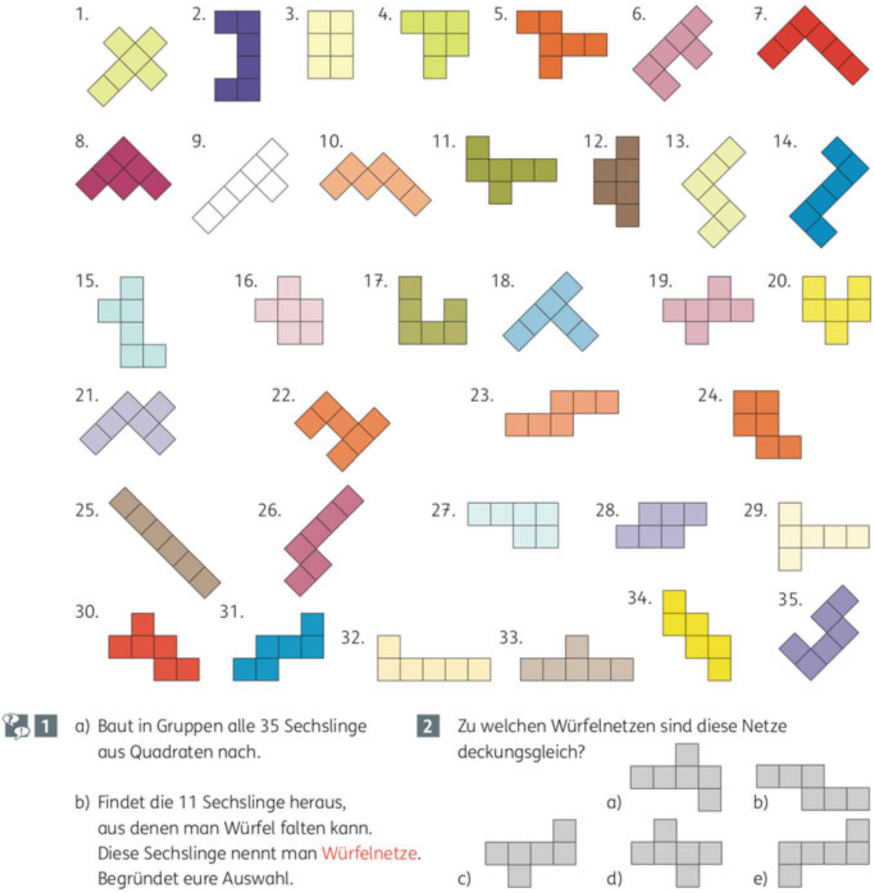
Susanne Prediger and Claudia Scherres have conducted guided clinical interviews with pairs of students in grade 5 (Prediger and Scherres 2012). The objective of this study has been to investigate in some depth how students proceed when trying to find as many different nets as possible. The authors applied quite a number of empirical instruments in order to obtain a differentiated picture of the processes occurring during the collaboration. The results of this study are very complex and therefore cannot be summarized in short terms. For the following comparison two findings are relevant (Prediger and Scherres 2012, p. 171):

1. Pairs of students can often exhaust their potential only through the intervention of the teacher.
2. The cooperation for exploiting the potential fully is enhanced when this cooperation is guided by mathematical considerations.

From the perspective of developmental research the first objective of a didactical analysis concerning the topic “nets of the cube” is to find out at which place of the curriculum students are in a position to respond to the requirements that certain treatment of this topic involves. At the very outset it should be kept in mind that any beautiful and important topic might allow for different approaches suitable for different places in the curriculum.

In the Mathe 2000 curriculum nets of the cube are embedded in the fundamental idea of “dissecting and recombining figures”, which is systematically developed along grade levels. An easy way of determining all possible nets is revealed in connection with polyominoes, a rich topic that was introduced by Golomb (1962) and elaborated for the primary level in Besuden (1984). A polyomino is a composition of congruent squares edge by edge. Polyominoes that are congruent are considered as equal. It is easy to see that there is only one domino (with two squares), but that there are two different triominoes (with three squares). Children in grade 3 easily find all 5 tetrominoes (with 4 squares) by adding one square to triominoes, and also all 12 pentominoes (with 5 squares) by entending tetrominoes. It is a stimulating task for kids to determine the 8 pentominoes that can be folded into an open cube.

In a textbook for grade 3, the 11 nets of a cube are obtained in the following way (Wittmann and Müller 2013, p. 65): The children are informed that it is possible to derive all 35 hexominoes by extending the 12 pentominoes. As this process would take too much time, the 35 hexominoes are provided by the teacher (Fig. 2) and the students are asked to find out which of these hexominoes are nets of a cube. In Fig. 2 the nets are arranged in five groups of 7 nets. This suggests forming five groups of students each of which has to make their 7 hexominoes with paper squares and sellotape and to investigate which ones can be folded into a cube. All five groups have to explain the reasons why some of their hexominoes do not produce nets. So in cooperation all 11 nets are determined through cooperation in a rigorous way.



1 a) Baut in Gruppen alle 35 Sechslinge aus Quadraten nach.

b) Findet die 11 Sechslinge heraus, aus denen man Würfel falten kann. Diese Sechslinge nennt man Würfelnetze. Begründet eure Auswahl.

2 Zu welchen Würfelnetzen sind diese Netze deckungsgleich?

Fig. 2 Selecting the nets of cubes from the set of hexominoes

An alternative approach at this level would be to start from the 8 pentominoes that can be folded into an open cube and to extend them to nets of a cube. However, as most nets can be derived from different nets of an open cube, it may be rather complicated to eliminate congruent nets.

In grade 5, the theme "nets of a cube" should be revisited. Again it seems appropriate to provide the students first with paper squares and sellotape and to stimulate them to find as many different nets as possible. Based on students' findings the teacher can guide the students to a systematic derivation of all possible nets. A natural way is to refer to the "addition principle" of combinatorics which consists of subdividing the set of combinatorial possibilities into subsets which are easier to manage. In the case of nets of the cube the maximum number of squares in a row is an appropriate criterion for a classification as is indicated briefly.

Case 1: 6 *squares in a row*
No cube is possible as there are overlays and two faces remain open.

Case 2: 5 *squares in a row*
Again no cube is possible as there is one overlay and one face remains open.

Case 3: *At most* 4 *squares in a row*
First it must be found out where a fifth square can be added so that a net becomes possible. For each of the two possible positions of the fifth square the possible positions of the sixth square have to be determined. Some care is needed to eliminate nets that are congruent to nets that have been found before. Figure 3 shows how to proceed stepwise starting from four squares in a row. The six nets determined in this way are drawn in bold lines.

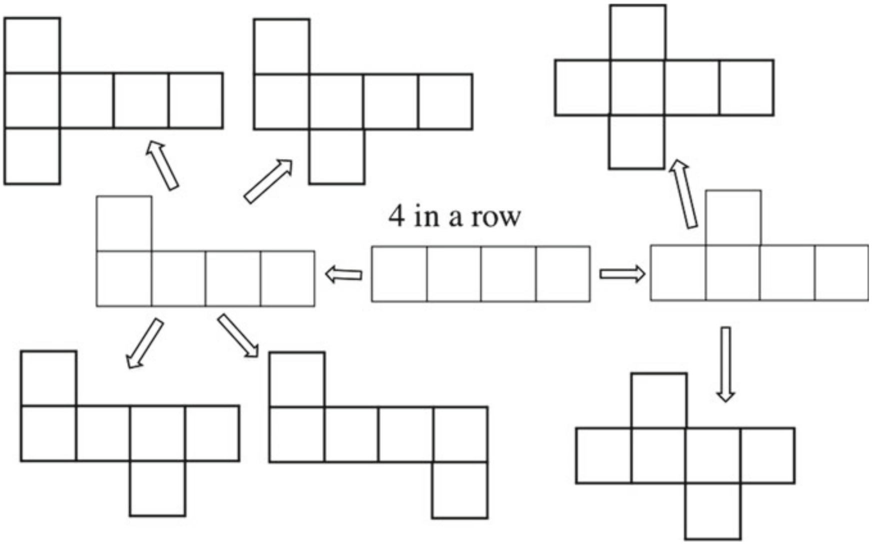


Fig. 3 Derivation of the nets of a cube where at most four squares are in a row

Case 4: *At most* 3 *squares in a row*
In Fig. 4 no arrows are drawn away from the four pentominoes on the right. The reason is that the extensions of these pentominoes would result in nets that were already found.

Case 5: *At most* 2 *squares in a row*
In this case there is essentially only one way to get a net (Fig. 5).

It is obvious that this systematic derivation of all 11 nets of the cube is not easy. However, only means are used that are accessible to students in grade 5. With the assistance of the teacher, this learning environment is good to handle.

Of course it cannot be predicted how the investigation of this learning environment might develop in a certain class. Every interaction takes place under the particular circumstances of the class. However, a teacher who knows the mathematical background

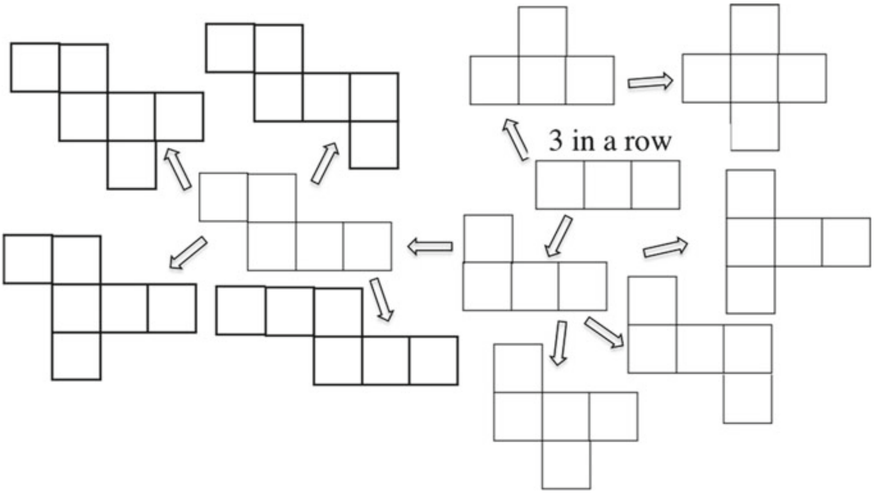


Fig. 4 Derivation of the nets of a cube where at most three squares are in a row

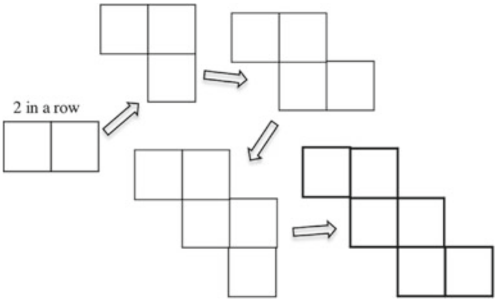


Fig. 5 Derivation of the only net of a cube where at most two squares are in a row

thoroughly is in a position to deal flexibly and productively with the contributions and ideas from the students. Based on their findings the teacher can introduce the classification. Different groups of students can investigate the three cases. In this way the complexity of the task is reduced to a reasonable level. The teacher can provide support where necessary.

Comparison: In this example the empirical investigation and the didactical analysis complement each other. Both are useful and instructive. There is no question that a teacher who has more insight into the processes linked to finding the various nets is more likely to interact with the students than a teacher who closely adheres to the mathematical structure and hardly leaves any room to the students. On the other hand, it is hard to imagine that a teacher who does not have a clear picture of the mathematical structure can organize a lesson solely with the spontaneous ideas of the students and with general pedagogical knowledge.

With respect to teaching and to teacher education, there are nevertheless significant differences between the two approaches. It is questionable if the “high resolution” instruments that have been employed in the empirical study by Prediger and Scherres (2012) can be communicated to teachers and students teachers in the time that is usually available in teacher education. It is also a question whether the results of this study can be integrated into teaching materials that work without the intervention of a teacher. The main findings prove the opposite.

In contrast, the didactical analysis requires only a relatively small amount of time and can be well integrated into teacher education. The language that is used is simple and easy to understand. If the nets of a cube are included in both mathematical and didactical courses in an inquiry-based way there is a good chance that the metacognitive and cooperative skills that have been found as important in the empirical study can be acquired implicitly in these courses. This, however, is not to devalue empirical studies. The aim of this paper is to plead for didactical analyses as one tool of mathematics education without excluding other tools.

4 Structure-Genetic Didactical Analyses

The approach of mathematics education *emerging from the subject* is based on the following assumptions:

1. Mathematical skills and techniques are acquired best in an active way under the guidance of mathematically experienced teachers. This refers to both teaching and teacher education. The practice of skills in its various forms plays a crucial role for successful learning.
2. The level of achievement that can be reached depends on the organization of teaching along fundamental mathematical ideas that are being revisited continuously. Only in this way is it possible to secure solid foundations for further learning and to brush up on prior knowledge. Also, only in this way it is also possible to provide mathematical structures as building blocks for modeling real situations. The development of curricula that are consistently and systematically designed accordingly and combine the orientation towards structures with the orientation towards applications is the central task of mathematics education.
3. Authentic mathematical activities in which heuristic plays a crucial role, are by their very nature social and communicative and include theories of teaching and learning *quite naturally* (implicit didactics). To make student teachers and teachers aware of these implicit theories by referring to their own mathematical experiences is the most direct and most efficient form of providing them with (explicit) didactical knowledge.

Against this background, didactical analyses as employed in the examples above are playing a fundamentally important role. This research method, which is the gold standard in mathematics education conceived of as a “design science”, is an extension of the traditional “subject matter didactics”. While the latter has been focused on the logical analysis of subject matter and too much linked to the "broadcast" method of transmitting knowledge from the teacher to the student, the extended method emphasizes both the genesis of knowledge over the grades and individual learning processes. In order to emphasize this wider perspective, the term *structure-genetic didactical analysis* is proposed for this extended method.

The above examples show that structure-genetic didactical analyses are linked to hard facts: to the mathematical practice in exploring, describing and explaining patterns on various levels, to the prerequisite knowledge of learners, to the objectives of teaching and to the curriculum. This is all *empirical material*. Therefore, the structure-genetic didactical analysis is an empirical method. Because of its nativeness it may well be considered as empirical research of "the first kind". The usual empirical studies are then empirical research of the "second kind". The assertion that only empirical studies of the second kind would provide "evidence-based models" for teaching and learning is untenable.

Structure-genetic didactic analyses are of primary importance in mathematics education for the following reasons:

1. They emerge from the *mathematical practice*, that is from *doing mathematics*, at various levels.
2. They foster an *active relationship* with mathematics as a living subject.
3. They are *constructive* and therefore absolutely essential for designing substantial learning environments and curricula.
4. They are natural guidelines for teachers, as they unfold the *implicit theories of teaching and learning* of mathematics, that is, as they "unfreeze" the "didactical moments frozen in the subject" (Heintel 1978, 46).
5. They are *meaningful for teachers*, as the feedback from the field clearly demonstrates.

The examples in the first three sections show that structure-genetic didactical analyses take the following points into account:

- mathematical substance and richness in activities at different levels,
- evaluation of the cognitive load on students,
- curricular matching (with respect to contents and general objectives)
- coherence and consistency along the curriculum,
- curricular reach,
- potential for practicing skills (most important!)
- estimation of the expenditure of time.

Paradigms of structure-genetic didactical analyses are Wheeler (1967), Freudenthal (1983) and the developmental research initiated by Hans Freudenthal at the IOWO in the 1970s, the developmental research initiated by Nicolas Rouche at the CREM in Belgium, see for example (Rouche et al. 1996), as well as the work of Heinrich Winter, the German Freudenthal, in particular (Winter 2015). These paradigms demonstrate that the development of mathematics education as a research discipline also depends on the design of conceptually founded substantial learning environments. Achievements in this direction have to be acknowledged as *results of research*.

In the context of this paper point 4 above is of particular importance and therefore deserves some elaboration. The idea that theories of teaching and learning are implicitly contained in the subject matter, and that therefore mathematics education is not completely dependent on imports of theories from other disciplines is by far not new. More than 100 years ago John Dewey has formulated this idea with a clarity that leaves nothing to be desired. In his paper there is a long enlightening section on the importance of the subject matter for teacher education (Dewey 1977, pp. 263–264):

> Scholastic knowledge is sometimes regarded as if it were something quite irrelevant to method. When this attitude is even unconsciously assumed, method becomes an external attachment to knowledge of subject-matter. It has to be elaborated and acquired in relative independence from subject-matter, and *then* applied.
>
> Now the body of knowledge which constitutes the subject-matter of the student teacher must, by the very nature of the case, be organized subject-matter. It is not a separate miscellaneous heap of scraps. Even if (as in the case of history and literature), it be not technically termed "science," it is none the less material which has been subjected to method—has been selected and arranged with reference to controlling intellectual principles. There is, therefore, method in subject-matter itself—method indeed of the highest order which the human mind has yet evolved, scientific method.
>
> It cannot be too strongly emphasized that this scientific method is the method of the mind itself. The classifications, interpretations, explanations, and generalizations which make subject-matter a branch of study do not lie externally in facts apart from mind. They reflect the attitudes and workings of mind in its endeavor to bring raw material of experience to a point where it at once satisfies stimulates the needs of active thought. Such being the case, there is something wrong with the "academic" side of professional training, if by means of it the student does not constantly get object-lessons of the finest type in the kind of mental activity which characterizes mental growth and, hence, the educative process. (...)
>
> Only a teacher thoroughly trained in the higher levels of intellectual method and who thus has constantly in his own mind a sense of what adequate and genuine intellectual activity means, will be likely, in deed, not in mere word, to respect the mental integrity and force of children.

For the teaching practice this view is of fundamental importance: The ancient Greeks understood '*theory*' as *view*. The Greek word for theory, $\theta\varepsilon\omega\rho\iota\alpha$ is derived from $\theta\varepsilon\omega\rho\varepsilon\iota\nu$, which means *viewing, regarding, observing*. In this original sense a *theory* provides a comprehensive view of some area that allows for acting purposefully in this area while taking some circumstances and contingencies in this area into account. The natural theories of teaching and learning embedded in subject matter serve exactly this purpose: they represent practicable theories *for the teacher*, and they supply him or her with profound information or knowledge on which to base her or his actions. Whether it is to introduce children to multiplication, or to practice long addition, or to determine the nets of the cube; or to estimate students' prerequisite knowledge, to activate their thinking, to interact and communicate with them; or to interpret students' oral and written utterings, to assess their learning progress or to start remedial work—all this is essentially determined by the teacher's "comprehensive view" of the topic to be learned. That teaching does not proceed smoothly, that there are breaks and obstacles in the learning processes, that students make mistakes, have difficulties in understanding some points, forget what they have learned before, and so on: This knowledge is an essential part of the implicit theories of teaching and learning arising from *an active mastery of subject matter*.

What therefore counts most in teacher preparation is not an explicit didactical component (i.e., method courses), but the *mathematical* component, *given that* in this component mathematical activities are offered that stimulate and provide student teachers with relevant experiences in regard to learning processes, including learning difficulties, phases of confusion, confidence in overcoming difficulties and so on.

Mathematical courses organized in this way also provide the most effective *theoretical* basis for teaching. This is not to say that theories imported from other disciplines are of no use. They may be. This is also not to say that method courses are superfluous. Rather, both imported theories and method courses can significantly enhance structure-genetic didactical analyses. However, they should not replace them.

5 Conclusion

This paper is a plea for structure-genetic didactical analyses, the empirical research of the first kind. It must not be misunderstood as a plea against empirical studies of the second kind. On the contrary, such studies are indispensible, when new topics are to be introduced, for which no information on students' prerequisite knowledge is available, and when new approaches or new means of representations are used. Examples are the introduction of stochastics at the primary level or the use of digital media. Empirical research of the second kind is also very useful for investigating the processes more closely that occur when a learning environment is "staged" in the classroom. Of course these studies are all the more revealing and more meaningful, the closer they are attached to structure-genetic analyses.

It has also to be acknowledged that a wider perspective in mathematics education including imports from related disciplines significantly contributes to a better understanding of mathematics and therefore supports structure-genetic didactical analyses. In this sense the present author has greatly profited from Jean Piaget's genetic epistemology. It is no accident that the term "genetic" is a constituent of the term "structure-genetic didactical analysis".

In a position paper on the nature of mathematics education Heinz Griesel contended that in his sense "didactical analyses" would not differ from the "logical analyses" of mathematics (Griesel 1974). Heinz Steinbring rightly rejected this narrow view (Steinbring 2011). With structure-genetic didactical analyses the situation is completely different. These analyses include logical analyses, it is true, however, they involve also knowledge about mathematical processes, about the curriculum, about students' prerequisite knowledge at different levels, and about the boundary conditions of teaching. A mere knowledge of (elementary) mathematics is by far not sufficient. To put oneself in the place of a child who takes his or her first steps in early mathematics, to look at the multiplication tab1e with the eyes of a second grader, to find the nets of a cube with the means that are available to students at the secondary level, or to make the concept of a limit accessible to high school students, all this requires a special didactical approach and a special sensitivity for the genesis of knowledge and for the mathematical practice at the level in question.

Mathematics education has certainly been enriched enormously by contributions from other disciplines. Structure-genetic didactical analyses are nevertheless the key for developing mathematics teaching and teacher education. Without them mathematics education is in danger to degenerate into a self-referential system. Jeremy Kilpatrick’s warning of the “reasonable ineffectiveness of research in mathematics education” should, thus, be taken seriously (Kilpatrick 1981).

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Chapter 14
Understanding and Organizing Mathematics Education as a Design Science–Origins and New Developments

Abstract The objective of this paper is

(1) to revisit briefly the conception of mathematics education as a design science as it has been evolving alongside the developmental research in the project Mathe 2000 from 1987 to 2012
(2) to report in some detail on recent developments, as concerns both conceptual and practical issues.

The paper is a plea for appreciating and (re-)installing "well-understood mathematics" as the natural foundation for teaching and learning mathematics.

Keywords Design science · Learning environments · Well-understood mathematics · Structure-genetic didactical analyses · Productive practice · Collective teaching experiments

1 Origins

It is no accident that in the 1960s, when the traditional content and methods of teaching mathematics were being questioned and there was a call for new content and methods, the very discipline that had been responsible for the teaching of mathematics for centuries, namely mathematics education (didactics of mathematics), was also questioned by a growing number of mathematics educators (didacticians) and considered to be no longer adequate.

Since at that time many mathematicians, among them prominent ones, were also committed to mathematics teaching, the discrepancy between the solid scientific foundation of mathematics and its missing counterpart in mathematics education was felt as painful, particularly by those, including the present author, who had moved from mathematics to mathematics education in order to specialize in this rapidly developing field.

In the subsequent discussions about the scientific status of mathematics education, the following questions were foremost:

(1) What is the relationship between mathematics and mathematics education?

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(2) What distinguishes mathematics education from mathematics?
(3) How can mathematics education establish a scientific basis that preserves its close and necessary connections with mathematics and that at the same time reflects its special mission with respect to the teaching practice and to teacher education?

At a conference organized by the Institute of Didactics of Mathematics (IDM) at the University of Bielefeld in 1975, Jeremy Kilpatrick made a crucial point when he distinguished between “theories imported from other disciplines” and “theories developed within mathematics education,” or “homegrown theories,” as he referred to the latter.

Since the early 1970s, the present author has been convinced that mathematics education would be far better served by “homegrown” theories, and so he has been looking for a framework appropriate for developing such theories. His idea to conceive of mathematics education as a design science was inspired by new developments in other fields and in mathematics education itself as will briefly be described in this section.

1.1 The Rise of the Sciences of the Artificial

In 1970, Herbert A. Simon, who in 1978 was awarded the Nobel Prize in economics, published a booklet in which he coined the term “design science” (Simon 1970). His intention was to delineate disciplines in which he was active (economics, administration, computer science, cognitive psychology) and disciplines like engineering from the established sciences and to provide these disciplines with a scientific *status of their own.* He identified the difference by highlighting “design” as the “principal mark” of design sciences (Simon 1970, 55):

> Historically and traditionally, it has been the task of the science disciplines to teach about natural things: how they are and how they work. It has been the task of engineering schools to teach about artificial things: how to make artifacts that have desired properties and how to design (…) Design, so construed, is the core of all professional training; it is the principal mark that distinguishes the professions from the sciences. Schools of engineering, as well as schools of architecture, business, education, law and medicine, are all centrally concerned with the process of design.

As “education” was mentioned explicitly here, it was only natural to consider mathematics education as a “design science.” The question, however, remained what the “artificial objects” of mathematics education might be. In Wittmann (1984), a proposal was made to consider “teaching units” as these artificial objects. Later this term was replaced by “learning environments.”

1.2 Developments in Management Theory

It is clear that there is a basic difference between a technical artifact, like a machine, which functions according to natural laws, and a "teaching unit" that cannot be used mechanically but requires the intelligent application by human beings as well as adaptation to the momentary social context. This difference is not restricted to education but is also typical of other design sciences, in particular economics.

In 1976, the Swiss management theorist Malik published a book in which he distinguished two classes of design sciences (Malik 1986):

- *Mechanistic-technomorph* design sciences based on the natural sciences
- *Systemic-evolutionary* design sciences dealing with complex systems that, in contrast with a machine, cannot be completely controlled from outside.

It was equally clear that mathematics education as a design science belongs to the latter class.

1.3 Prototypes of Design in Mathematics Education

The discussion about mathematics education in the 1970s was also very much influenced by fresh contributions to mathematics education that transcended the traditional scope.

In the preface of his book "Basic Notions in Algebra," the eminent Russian mathematician Igor Shafarevics (1989, 4) states:

> The meaning of a mathematical notion is by no means confined to its formal definition; in fact, it may be rather better expressed by a (generally fairly small) sample of the basic examples, which serve the mathematician as the motivation and the substantive definition, and at the same time as the real meaning of the notion.

In the same sense, typical projects in developmental research explain the notion of mathematics education as a design science much better than general descriptions.

In 1965 and 1967, two groups of English mathematics educators published books that consisted of the description of teaching ideas and teaching units combined with the explanation of the mathematical background as well as hints for teaching (Fletcher 1965; ATM 1967).

The same approach was pursued on a larger scale at the Dutch Instituut voor Ontwikkeling Wiskunde Oderwijs (IOWO), founded in 1971 under the direction of Hans Freudenthal. A good summary of the developmental research conducted at the IOWO is provided by Freudenthal et al. (1976).

A third important impetus came from Japan. Here again, the progress in mathematics education was communicated by means of carefully formulated teaching units (Becker and Shimada 1997).

Last not least, the work of Heinrich Winter, the "German Freudenthal", must be mentioned as a major inspiration. In Winter's seminal paper on general objectives of

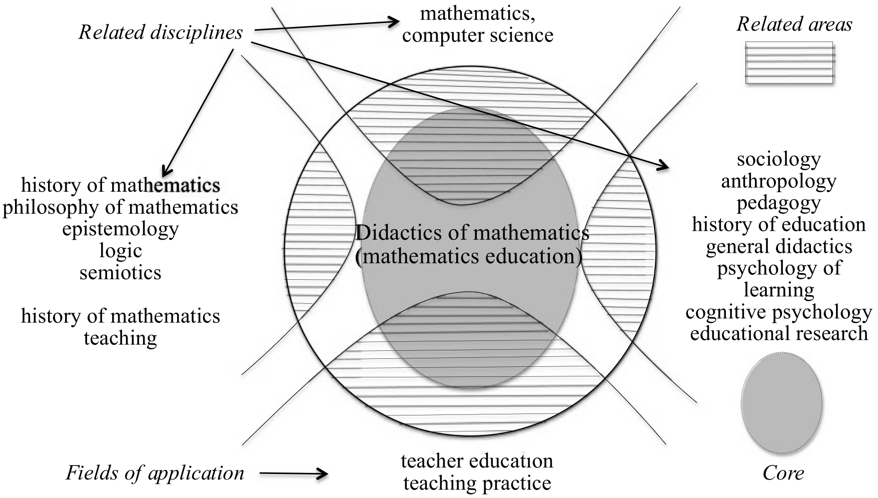

Fig. 1 Mathematics education as a design science and its interdisciplinary relationships

mathematics teaching, these objectives were illustrated by teaching examples (Winter 1975).

1.4 *The Map of Mathematics Education as a Design Science*

In Wittmann (1995, 89), the conception of mathematics education as a design science was summarized in a diagram that is shown with some modifications in Fig. 1.

The *core* of mathematics education (didactics of mathematics) represents the design, the empirical research and the implementation of learning environments. It is surrounded by "*related disciplines*" and "*fields of application*." The related areas are the intersections of mathematics education with the related disciplines.

In Wittmann (2001), this conception was elaborated on with respect to systemic constraints.

The map in Fig. 1 was intended to give a foundation to the developmental research undertaken in the projects mentioned in Sect. 1.3. However, as it has turned out, this map can also be interpreted differently. As the related disciplines on the right side of the diagram have offered and continue to offer advanced theories of teaching and learning as well as theories on the educational system's societal background, it has been and continues to be tempting for mathematics educators to take these theories as starting points for establishing a scientific basis of mathematics teaching. In the decades since, the mainstream of mathematics education has been moving in this direction to the extent that this approach now enjoys a near monopoly. As this movement has more or less been ignoring the tradition of mathematics education in a similar way in which “New Math” has ignored the tradition of mathematics teaching, the present author suggests calling it “New Math Education.” Although “New Math Education” has widened and refined the scope of research and led to progress in many fields, it is clear that it has weakened the connections of mathematics education with both mathematics and the teaching practice.

In recent years, an attempt has been made to include “design” in “New Math Education”. The result has been termed “design research” (Cobb et al. 2003; Prediger 2015). However, this direction of research, which follows the paradigms of applied science, differs from what was originally intended by conceiving of mathematics education as a design science.

In order to make the difference explicit, it is necessary to highlight the singular role of mathematics among the related disciplines in Fig. 1.

2 Conceptual Developments

The 25th anniversary of the project Mathe 2000 was the right time to rethink the concept on which this project had been based and to formulate a revised concept for further research. It turned out that it was precisely those questions that had been addressed in the discussion about the scientific status of mathematics education in the 1970s that had to be taken up again (see Sect. 1.1 above). The answers at which we have arrived will be described in this section. In Sect. 3, some practical consequences will be illustrated by means of typical examples from the project.

2.1 The Natural Theory of Teaching: “Well-Understood Mathematics”

There is no disagreement that mathematics provides the subject matter of teaching and that therefore teachers must “know mathematics” in order to teach the subject properly. A closer look at the problem, however, reveals that there are quite different interpretations of this general statement and that, as a consequence, there are quite different views of the roles that mathematics should play in mathematics education and in teacher education.

At the 1975 Bielefeld conference, John LeBlanc described the basic issue in full clarity (LeBlanc 1975):

> The content of many mathematics courses was felt to be irrelevant to many of the prospective teachers. The new requirements for the preparation of elementary teachers left mathematics departments looking for materials appropriate for such courses. At the same time, the mathematicians selecting the books were also under some pressure to make sure that the content was mathematically honest. Few, if any, materials existed that met both criteria of educational appropriateness and of mathematical honesty. The latter requirement usually was the winning criterion. The effect of inappropriate but mathematically honest materials

> was often just the opposite of that which was desired. The prospective teachers seemed to be even less confident than ever in mathematics and their attitude toward it became increasingly negative.

This description might well apply to the present situation in many parts of the world, particularly in the U.S. The "math war" in this country was definitely driven by different views held by mathematics educators and mathematicians. The books by Jensen (2003) and Wu (2011), both published by the American Mathematical Society, represent the intention to be mathematically honest and suggest that this kind of mathematics is not only necessary but also sufficient for teaching mathematics properly at the elementary level.

However, as early as in 1986, this view was fundamentally challenged by Lee Shulman in a seminal paper in which he contrasted mere "content knowledge" with "pedagogical content knowledge" and "curricular knowledge" (Shulman 1986). His proposal to look at content in a comprehensive way was taken up and elaborated on in mathematics education in a series of papers (see, for example, Ball et al. 2008).

In the European and Asian contexts, this broader view on content has always been present in teacher education, including teacher education for the elementary level. So it was only in the context of the U.S. that the book by Liping Ma (1999) could be presented as a revelation.

Our attempts in the project Mathe 2000 to better understand the impact of mathematics on mathematics education were greatly influenced by three papers that John Dewey, one of the greatest minds of all time in the area of education, had published as early as 1903–1904.

In Dewey (1903a, 285), an important distinction is made between two different views of a subject:

> Every study or subject has two aspects: one for the scientist as a scientist, the other for the teacher as a teacher (…) For the scientist the subject-matter represents simply a given body of truth to be employed in locating new problems, instituting new researches, and carrying them through to a verified outcome. To him the subject-matter of the science is self-contained (…) He is not, as a scientist, called upon to travel outside its particular bonds. (…) The problem of the teacher is a different one. As a teacher he is not concerned with adding new facts to the science he teaches. (…) He is not concerned with the subject matter as such, but with the subject matter as a related factor in a total and growing experience. Thus to see it is to psychologize it.

The connections and the differences between logically and psychologically organized subject matter were clarified in another eye-opening paper of Dewey's in great detail where Dewey arrived at the following conclusion (Dewey 1903b, 227–228):

> The serious problem of instruction in any branch is to acquire the habit of viewing in a twofold way which is taught day by day. It needs to be viewed as a development *out* of the present habits and experiences of emotion, thought, and action; it needs to be viewed also as a development *into* the most orderly intellectual system possible. These two sides, which I venture to term the psychological and the logical, are the limits of a continuous movement rather than opposite forces or even independent elements.

In Dewey (1904), a whole chapter is devoted to the role that courses on the subject matter of teaching should play in teacher education. Dewey insists on seeing subject matter not only as a fixed body of knowledge but also as a developing process that involves methods of teaching as well (Dewey 1904, 263–264):

> Scholastic knowledge is sometimes regarded as if it were something quite irrelevant to method. When this attitude is even unconsciously assumed, method becomes an external attachment to knowledge of subject-matter. It has to be elaborated and acquired in relative independence from subject-matter, and *then* applied.
>
> Now the body of knowledge which constitutes the subject-matter of the student teacher must, by the very nature of the case, be organized subject-matter. It is not a separate miscellaneous heap of scraps. Even if (as in the case of history and literature), it be not technically termed "science," it is none the less material which has been subjected to method—has been selected and arranged with reference to controlling intellectual principles. There is, therefore, method in subject-matter itself—method indeed of the highest order which the human mind has yet evolved, scientific method.
>
> It cannot be too strongly emphasized that this scientific method is the method of the mind itself. The classifications, interpretations, explanations, and generalizations which make subject-matter a branch of study do not lie externally in facts apart from mind. They reflect the attitudes and workings of mind in its endeavor to bring raw material of experience to a point where it at once satisfies and stimulates the needs of active thought. Such being the case, there is something wrong with the "academic" side of professional training, if by means of it the student does not get constantly get object-lessons of the finest type in the kind of mental activity which characterizes mental growth and, hence, the educative process. (…) Only a teacher thoroughly trained in the higher levels of intellectual method and who thus has constantly in his own mind a sense of what adequate and genuine intellectual activity means, will be likely, in deed, not in mere word, to respect to the mental integrity and force of children.

Shulman (1986, 6–7) elaborates on the fact that in antiquity and in the Middle Ages there was no distinction between research and teaching. It is no accident that the term "mathematics" is derived from the Greek $\mu\alpha\theta\eta\mu\alpha\tau\iota\kappa\eta\ \tau\epsilon\chi\nu\eta$ (mathematike techne), which denotes the "art of teaching and learning." Even in modern Greek, $\mu\alpha\theta\alpha\iota\nu\omega$ (mathaino) means "learning."

Later on, "knowledge" and "learning" became more and more separated. However, at the forefront of research, the connection between research and teaching has always been close, right up to the present day. William Thurston, who was awarded the Fields Medal in 1982, defended his use of broader means of representation with his intention to support understanding (Thurston 1994, 162):

> It may sound almost circular to say that what mathematicians are accomplishing is to advance human understanding of mathematics (…) If what we are doing is constructing better ways of thinking, then psychological and social dimensions are essential to a good model for mathematical progress.

From these descriptions we have drawn the following conclusion: If mathematics is understood and practiced as a living and developing organism including guiding problems, heuristic strategies for solving problems, different types of representation (enactive, iconic, symbolic), different ways of communication, exercises at different levels, the search for structures and patterns, proofs, and applications to internal or real-world problems, then the organization of knowledge in order to make it understandable to students is part and parcel of this "well-understood mathematics" *at any level*, beginning with early math (Kinnear and Wittmann 2018).

Mathematics has grown historically, and this growth provides a good starting point for appreciating "well-understood mathematics." This is not to say that teaching should follow the historical order. However, history gives valuable information about how to develop mathematics *genetically* and shows that the lower stages of development are *indispensable* for the higher stages and must be appreciated in their *specific manifestations*. It is a fundamental mistake to believe that formal mathematical analyses of the subject matter can replace elementary formulations (see Dewey 1903b). "Mathematical honesty" must not be reserved for formal analyses, which nevertheless are indispensable as an orientation for the development of coherent curricula (see Sect. 2.2).[1]

If mathematics education is based on "well-understood mathematics," then there is no compelling reason to look *exclusively* for theories of teaching and learning outside of mathematics in order to secure a scientific basis. *The natural theory of teaching and learning mathematics is implicit in "well-understood mathematics*." The literature on elementary mathematics offers a veritable goldmine for "well-understood mathematics" waiting to be exploited in the design of learning environments and in teacher education. Research on the history of mathematics and on the philosophy of mathematics is of great help in elaborating on this natural theory.

2.2 Structure-Genetic Didactical Analyses

In traditional mathematics education, didactical analyses have been the main method for shaping conceptions for teaching certain areas. This method is taken up in the design science approach but further specified in the following way: Subject matter is considered *in its development* with respect to the development of learners at different levels. Both "mathematical honesty" and "educational appropriateness" are taken seriously and brought to a natural synthesis. In order to express this extended method properly, the term "structure-genetic didactical analyses" has been coined (Wittmann 2018).

Structure-genetic didactical analyses represent a "bottom-up" view of teaching and learning that fundamentally differs from a "top-down" view. In this respect, Freudenthal's critique of Chevallard's demand for a "didactical transposition" from the knowledge of scholars to the school environment is enlightening (Freudenthal 1986, 326–327).

Structure-genetic didactical analyses offer important advantages (Wittmann 2018, 145):

1. They emerge from mathematical practice, that is, from doing mathematics at various levels.
2. They foster an active relationship with mathematics.

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[1] In the eyes of the present author, U.S. mathematicians lost the "math war" because of their inappropriate top-down perspective. It is a pity that they failed to refer to "well-understood mathematics.".

3. They are constructive and therefore absolutely essential for designing substantial learning environments and coherent curricula.
4. They are natural guidelines for teachers, as they bring to fruition the implicit theories of teaching and learning mathematics and "unfreeze the didactical moments frozen in the subject" (Heintel 1978, 46).

Point 3 is most important, as success in learning greatly depends on linking new knowledge to old knowledge in a coherent way. This is in line with David Ausubel's famous statement (Ausubel 1968):

> The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly.

What the learner knows, however, is mainly determined by prior learning. Therefore, a coherent and consistent curriculum in which care is taken to establish solid knowledge *continuously* plays a decisive role in teaching and learning. In order to design such curricula a thorough knowledge of mathematics *across all levels*, including logical analyses, is crucial.

2.3 A Differentiated Conception of Practicing Skills

While Sects. 2.1 and 2.2 are concerned with connecting mathematics education to mathematics, that is, to "well-understood mathematics," this subsection and the next one are devoted to linking mathematics education to teacher education and to the teaching practice.

One of the oldest principles of learning is summarized in the Latin saying, "Repetitio mater studiorum" (practice makes perfect). While this principle is unquestioned in fields like music or sports, it is often woefully neglected when it comes to learning. It is indicative that in contemporary mathematics education, notably in "New Math Education," "practice" is hardly a topic of research. In the Western world, "practice" is most commonly understood as "drill and practice" and therefore rejected in principle, in marked contrast with Asia. However, even teachers in Western countries know that without extensive "practice," no real and lasting progress is possible.

The Mathe 2000 project made deliberate attempts to overcome the seeming contradiction between "practice" and "understanding" or between "genuine mathematics" and the "basics." The result was a differentiated conception of practicing skills. In Wittmann & Müller (2017, 141) a distinction is made between three types of practice:

"*Introductory practice*" aims at making students familiar with a new topic, that is with new problems, new means of representation, new vocabulary, new symbols, new methods, etc. The main objective is to firmly link new knowledge to prior knowledge. According to Wittgenstein's language game, students acquire new knowledge through its *repeated* use in meaningful contexts.

"*Basic practice*" refers to the extended practice of a small set of skills that occur frequently, the so-called *basic competences*, which must be mastered *automatically*.

"*Productive practice*" is a kind of magic wand: It integrates the practice of skills with the exploration and explanation of patterns, with the solution of problems and with applications. The term "*productive practice*" was coined by Heinrich Winter, who wanted to emphasize that students are expected to "produce" something on their own in this type of practice. *Productive practice* represents the view of mathematics as the science of patterns in full manifestation, and, at the same time, it offers a number of practical advantages (little preparation on the part of teachers, self-monitoring on the part of students, natural differentiation between students, time saved).

2.4 Awareness of Systemic Constraints

It is a tacit assumption in "New Math Education" that teaching and learning represent realities that can be investigated and controlled roughly in the same way as physical realities. This assumption is a mere fiction. Of course, there are *momentary* local realities of teaching and learning. However, these are not given and enduring but rather shaped by former (formal or informal) teaching and learning, in whatever form this may have taken place, and they are fluid and shaky. Donald Schön has convincingly shown that in complex systems, the methods of "applied science" are of limited value and that therefore the professionals in these fields must make decisions on their own in their local environment (for more details see Wittmann 2016).

As it is the communication with teachers that counts, it has been a conscious decision in our project to provide them with a robust theory for teaching that can be communicated in understandable language and to empower teachers to act in a self-reliant way. Here again, the design science approach and "New Math Education" differ fundamentally. The results of research in "New Math Education" are extremely diverse and formulated in a technical language. Their sheer mass is excessive, and it is hard to imagine how they can reach the practice of teaching.

The most promising way for taking systemic constraints into account seems to be introducing teachers to "well-understood mathematics" and connecting it to substantial learning environments. In our view, it is first-hand knowledge in this area that is the most important asset in terms of professional knowledge. This knowledge greatly facilitates work with students, including communication and social interaction, and it provides teachers with the necessary flexibility in subject matter that is needed for meeting the demands of individual students.

The natural theory of teaching and learning as implicit in "well-understood mathematics" might appear "naïve" in comparison to the grand theories offered by "New Math education." However, from the systemic perspective this is a decisive advantage.

From this perspective, there is another important point: Although international exchange in mathematics education has made huge progress compared to the pre-1970s situation, its impact on schools can only be effective at a *local* level. In this respect, the involvement of teachers is crucial (Fung 2016).

3 Practical Consequences

In this section, some practical consequences of the new conceptual basis will be illustrated by means of examples that are taken from the new "Handbook of Practicing Skills in a Productive Way" (Wittmann and Müller 2017/2018). The choice of the title was a deliberate decision with respect to our systemic credo. In fact, this handbook is not just about practice but rather offers a comprehensive introduction to teaching arithmetic in the first four years based on "well-understood" mathematics.

The following four subsections are only loosely linked to the subsections in Sect. 2 as the various points discussed in this section overlap and, as a rule, each learning environment exemplifies several of them.

3.1 Integrating "Well-Understood Mathematics"

In a letter submitted to the working group on proof at ICME 7, Québec 1992, Yuri I. Manin introduced the term "mathscape" for the mathematical landscape research mathematicians see in their mind's eye and explore.[2] It is only natural to combine this metaphor with the term "learning environment" and, in a further step, to compare the role of the teacher with the role of a mountain guide. The job of the latter is to select tours that are appropriate for a certain group of hikers and to guide them to certain summits. In order to act professionally, the guide must have *first-hand experience* of mountainous landscapes, know how demanding certain tours are and have options for changing or abridging tours it if this should be advisable or even necessary. In a similar way, teachers must have first-hand experience of the "mathscape" on which a learning environment is based, and they must have different hiking options at their disposal.

In the new handbook, the subject matter is ordered in chapters devoted to subject areas. For each area, several learning environments are offered. Each chapter starts with an introduction into the mathematical structure of the area in question. The description of each individual learning environment also begins by laying out the mathematical structure of this environment in more detail. Teachers are invited to *first* "go on tour" *for themselves* in order to become familiar with the "mathscape" involved.

Example: The Learning Environment "Guessing Dice"

This environment is part of the subject area "Productive practice of addition and subtraction" in grade 1 (Wittmann and Müller 2017, 125–127).

The guiding problem is as follows: The teacher rolls three dice behind a barrier and announces only the total of the three numbers. In order to find out which numbers were rolled, the children are allowed to ask questions which can be answered with "yes" or "no," for example, "*Is there a* 6?" or "*Is there a* 4?"

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[2] Manin's letter is reprinted in Wittmann 2002b, 546.

Table 1 Partitions of the numbers 3 to 18 in three parts ≤ 6.

Total	Decompositions
3	1+1+1
4	2+1+1
5	3+1+1,2+2+1
6	4+1+1,3+2+1,2+2+2
7	5+1+1,4+2+1,3+3+1,3+2+2
8	6+1+1,5+2+1,4+3+1, 4+2+2,3+3+2
9	6+2+1, 5+3+1,5+2+2,4+4+1,4+3+2 3+3+3
10	6+3+1 , 6+2+2, 5+4+1, 5+3+2, 4+4+2, 4+3+3
11	6+4+1 ,6+3+2,5+5+1,5+4+2,5+3+3, 4+4+3
12	6+5+1, 6+4+2, 6+3+3, 5+5+2, 5+4+3, 4+4+4
13	6+6+1, 6+5+2, 6+4+3, 5+5+3, 5+4+4
14	6+6+2, 6+5+3, 6+4+4,5+5+4
15	6+6+3, 6+5+4, 5+5+5
16	6+6+4 ,6+5+5
17	6+6+5
18	6+6+6

Any answer provides additional information so that the numbers can be determined step by step.

Like all environments for *productive practice*, this learning environment requires a certain mastery of the skills that are involved—in this case, the addition and subtraction tables. However, these skills are applied in various ways and thus corroborated and consolidated.

For some totals (for example, 3, 4, 17, 18), no questions are needed, as there is only one triple of numbers with this total. For other totals there are up to six triples of numbers. Table 1 shows the combinatorial possibilities. These are called *partitions*, as the order of the summands does not matter. In order to avoid multiplicity, it makes sense to write the three summands in decreasing order. Listing the partitions in lexicographic order is a way to determine them systematically. This method can also be applied to determining partitions for which the size of the biggest part is not restricted, unlike in this case, where the limit is 6.[3]

Teachers who are familiar with the structure in Table 1 will be well prepared for guiding children through this learning environment.

In order to activate the readers of the handbook beyond the mathematics in the learning environments, each chapter ends with a section called "Search and Find for the Reader." The problems that are offered for investigation here use only the mathematics and means of representation of this chapter and so contribute to enhancing

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[3] *Partitions* are well suited as a field of study in mathematical courses for teachers.

the professional knowledge of teachers. Although the problems are somewhat more demanding, they are nevertheless accessible experimentally and can also be tackled by talented students.

Example:
In the chapter “Introduction to the Thousand Space,” digit cards for the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are used as one of the standard teaching aids. The problem in the corresponding section “Search and Find for the Reader” is as follows:

Take the nine digit cards

and arrange them into three 3-digit numbers so that

(a) the difference between the biggest and the smallest number is as big as possible,
(b) the difference between the biggest and the smallest number is as small as possible,
(c) the difference between the biggest and the middle number and the difference between the middle number and the smallest number are equal.
(d) Try to make the difference in (c) as small as possible.

Hint: The smallest possible difference is smaller than 100.

3.2 Designing a Consistent and Coherent Curriculum

In the revision of the handbook, structure-genetic didactical analyses were used both globally and locally.

Local analyses were applied for designing “mathscapes” that invite students (and teachers) to mathematical activities (see the examples in 3.1 and 3.3).

The dominant *global* objective was to design a consistent and coherent curriculum. “Consistent” means that the language, the means of representation and the problems to be addressed should fit together over the grades. “Coherent” means that there should be a seamless sequence of learning environments that build on one another.

The main instrument for achieving curricular consistency and coherence was the following list of seven fundamental ideas of arithmetic that allow for a genetic development of the subject matter (Wittmann and Müller 2017, 144):

1. Number as a synthesis of the ordinal and the cardinal aspect
2. Arithmetical laws
3. The structure of the decimal system
4. Algorithms
5. Arithmetical patterns
6. Numbers in the environment
7. Applications

Special attention was paid to the idea in No. 2, as Heinrich Winter’s demand for an “algebraic penetration of arithmetic” should be put into practice. To this end, the arithmetical laws had to be introduced *as early as possible*. Addition and subtraction presented no problem as the associative and the commutative laws of addition can easily be based on operations with counters. For multiplication and division, we chose rectangular arrays of counters and dots for the simple reason that this representation is the *only one* which allows for establishing the associative and commutative laws of multiplication as well as the distributive law at an elementary level (Freudenthal 1983, 109).

In Wittmann and Müller (2017, 71, 202–204), operative proofs of the five laws are presented that rest on the following invariance principle: The cardinal number of a set of counters (or dots) is independent of the location of the counters (dots).

The proofs run as follows:

Addition means that two sets of counters are united to form one set. Whether this operation is executed in one or several steps does not affect the result. In algebraic formulation: $a + (b + c) = (a + b) + c$.

Also the result does not depend on the order in which the two sets are put together: $a + b = b + a$.

The commutative law of multiplication is easily derived from the fact that rows and columns in a rectangular array $a \cdot b$ of dots change roles when the array is rotated by 90°. No dot is taken away, no dot is added. Therefore $a \cdot b = b \cdot a$.

Any array $a \cdot b$ can be separated into two arrays by means of a vertical or a horizontal segment or into four arrays by means of a horizontal and a vertical segment. In this way, the distributive law is established:

$$a \cdot (b_1 + b_2) = a \cdot b_1 + a \cdot b_2, \quad (a_1 + a_2) \cdot b = a_1 \cdot b + a_2 \cdot b$$

$$(a_1 + a_2) \cdot (b_1 + b_2) = a_1 \cdot b_1 + a_1 \cdot b_2 + a_2 \cdot b_1 + a_2 \cdot b_2$$

If we take an array $a \cdot b$ and arrange c copies of it consecutively, we get a large array with $c \cdot (a \cdot b) = (a \cdot b) \cdot c$ dots. As this array has a rows and $c \cdot b = b \cdot c$ dots in each row, we get $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Table 2 gives an overview of the curricular structure of arithmetic for grades 1 to 4 at which we have arrived. The coherence of the topic areas is expressed with arrows.

The table contains bars running from left to right that indicate the continuous development of the two fundamental ideas “Decimal system” and “Arithmetical laws” over the grades.

In addition, there is a third bar: “Calculightning” (in German “Blitzrechnen”). This word is an artificial combination of the words “calculating” and “lightning” and denotes a course with 40 basic competences that have to be mastered so they can be performed automatically (see Table 3).[4]

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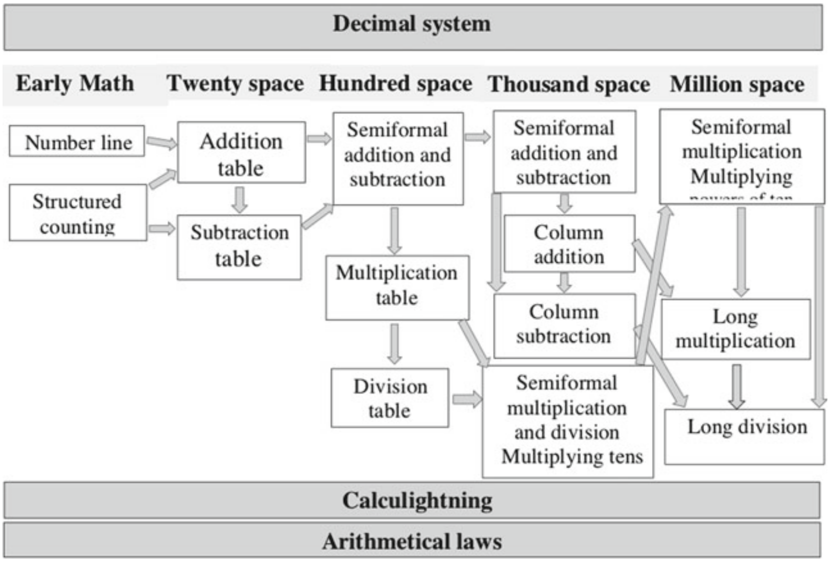
[4] “Calculightning” (“Blitzrechnen”) is available in the form of four apps (corresponding to grades 1–4) that offer also an option for English.

Table 2 The curricular structure of arithmetic in grades 1–4



As the names of the modules indicate, these competences are not independent but build on one another, both horizontally across the number spaces and vertically within each number space.

“Calculightning” is important for two reasons: It secures not only a firm mastery of a small set of basic competences, but it also serves as a remedial program for students who need additional support. At first glance, these two objectives might appear contradictory. However, a closer look at “Calculightning” reveals that this course which is derived from the fundamental ideas of arithmetic contains exercises that are crucial for understanding the decimal system and for establishing connections between number facts. Moreover, all modules of “Calculightning” are introduced in the context of *introductory practice*, which aims at facilitating understanding. Automation then comes in as the very final step in mastering these basic competences.

For the multiplication table this means:

This table is introduced by means of rectangular arrays of dots, whereby the connections provided by the arithmetical laws are used for effective learning.

It is only after extensive *introductory practice* that *basic practice* of the table begins with the ultimate goal of automation.

Learning environments for *productive practice* are investigated in parallel to this and include operative proofs based on rectangular arrays of dots (for details see Wittmann and Müller (Wittmann and Müller, 2017), Chaps. 4–6).

Table 3 Overview of the course “Calculightning”

Twenty Space	Hundred Space	Thousand Space	Million Space
How many?	how many? Which number?	Multiplication and division table	Reading and writing big numbers
Row of twenty	counting in steps	Doubling /Halving in the hundred space	Complementing to 1 million
Power of five	complementing to the next ten	How many? which number?	Dividing powers of 10in equal parts
Decomposing	complementing to 100	Counting in steps	subtracting powers of ten
Complementing to 10/20	Dividing 100 in equal parts	Complementing to 1000	Reading numbers differently
Doubling	Doubling/halving	Dividing 1000 in equal parts	Counting in steps
Addition table	Easy addition problems	Doubling/Halving in the thousand space	Doubling/halving in the million space
Subtraction table	Easy subtraction problems	Easy addition and Subtraction problems	Easy addition and Subtraction problems

3.3 *Including Operative Proofs*

Proofs are “the very heart of mathematics” (Günter Ziegler). At lower levels, it is appropriate to include “operative proofs” that use mathematical structures integral to informal means of representation (Wittmann 2002, 545–548). The conception of *productive practice* is well suited to combine the practice of skills with mathematical investigations, including proofs, and thus reflects “well-understood mathematics” in a particularly significant way.

A typical example is provided by the learning environment “ANNA numbers” (Wittmann and Müller 2018, Sect. 2.2.3). In this environment, the practice of column subtraction is combined with a mathematical investigation.

ANNA numbers are 4-digit numbers like 4114, 7887, 3003, etc. For any pair of different digits, there are two ANNA numbers, and the smaller number of each pair can be subtracted from the larger one: $4114 - 1441, 8778 - 7887, 3003 - 0330$, etc.

During the calculations, the following patterns emerge and can be discovered by students (and teachers):

(a) Only the results 891, 1782, 2673, 3564, 4455, 5346, 6237, 7128, and 8019 are possible, and they show conspicuous patterns.
(b) ANNA numbers with the same difference of digits have the same result.
(c) All results are multiples of the smallest result 891.

One proof of these patterns uses the place value chart (Fig. 2) and runs as follows: Usually subtraction is defined as “taking away.” A second, mathematically more advanced interpretation is “complementing.” Here $a - b$ means finding the number

Fig. 2 Operative proof based on the place value chart

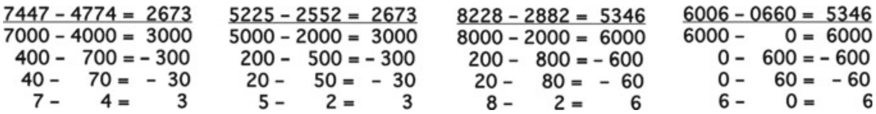
c which added to b yields a. In other words: b is complemented by c to yield a, and c is the difference $a - b$.

The left place value chart in Fig. 2 shows how the number 1221 is complemented to equal 2112. One counter is moved from the tens column to the ones column, and one counter from the hundreds column is moved to the thousands column. This operation increases the number 1221 by $+1000 + 1 - 100 - 10 = 891$.

In the place value chart on the right side of Fig. 2, two pairs of counters must be moved correspondingly. Therefore, the difference between 3113 and 1331 is $2 \cdot 891 = 1782$.

In order to go from 4994 to 9449, five pairs of counters must be moved. The difference is $5 \cdot 891 = 4455$. Clearly, the difference of the digits determines how many pairs of counters must be moved.

A second proof uses the semiformal strategy "separate place values" that is introduced as early as grade 2 with 2-digit numbers (Fig. 3).[5]



7447 - 4774 = 2673	5225 - 2552 = 2673	8228 - 2882 = 5346	6006 - 0660 = 5346
7000 - 4000 = 3000	5000 - 2000 = 3000	8000 - 2000 = 6000	6000 - 0 = 6000
400 - 700 = - 300	200 - 500 = - 300	200 - 800 = - 600	0 - 600 = - 600
40 - 70 = - 30	20 - 50 = - 30	20 - 80 = - 60	0 - 60 = - 60
7 - 4 = 3	5 - 2 = 3	8 - 2 = 6	6 - 0 = 6

Fig. 3 Operative proof based on the semiformal strategy "separate place values".

The calculations show that the results depend only on the difference of the digits and that all results are multiples of the smallest result $1000 - 100 - 10 + 1 = 891$.[6]

A third operative proof starts with the fact that a difference remains unchanged if both the minuend and the subtrahend are increased by the same number.

Starting from $1001 - 0110 = 891$ and increasing both digits by 1 step by step increases both numbers by 1111 and leads to $2112 - 1221$, $3223 - 2332$, $4334 - 3443$, etc. All these differences have the same result of 891.

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[5] The calculations run as follows: First the problem is written down. Then a line is drawn, and under it, the subtractions for the different place values are executed. Finally, the partial results are mentally combined for the final result, which is then entered in the first line.

[6] This proof is close to the algebraic proof where a pair of ANNA numbers is represented by $A \cdot 1000 + B \cdot 100 + B \cdot 10 + A \cdot 1$ and $B \cdot 1000 + A \cdot 100 + A \cdot 10 + B \cdot 1$, $A > B$. The difference of the two numbers is $(A - B) \cdot (1000 - 100 - 10 + 1) = (A - B) \cdot 891$.

Starting from 2002 − 0220 = 1782 and increasing the digits by 1 step by step leads to 3223 − 2332, 4334 − 3443, etc. Again, the result 1782 does not change.

In an analogous way, we can start with 3003 − 0330 = 2673 or 4004 − 0440 = 3564 etc. In all cases, the results remain invariant.

In order to show that all results are multiples of 891, we start from 1001 − 0110 = 891 and pass over to 2002 − 0220, 3003 − 0330, …, 9009 − 0990. At each step, the minuend increases by 1000 + 1, the subtrahend by 100 + 10. Therefore the difference grows by 1000 + 1 − 100 − 10 = 891. So 1728 = 891 + 891, 2637 = 1782 + 891, etc.

A fourth operative proof, which is preferable in the context of practicing column subtraction, rests on an analysis of the subtraction algorithm. In Germany the "complementing method" is still widespread (although unfortunately the mathematically less advanced method common in English-speaking countries is gaining ground). Figure 4 shows some calculations according to the "complementing method." As the name indicates, this method consists of complementing the subtrahend such that the subtrahend and the complement add up to the minuend. The notation is minimalistic, and the connection with column addition is obvious.

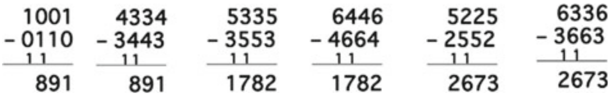

Fig. 4 Operative proof based on column subtraction (complementing method)

In all results of Fig. 4, the tens digit and the ones digits add up to 10, the hundreds digit is 1 less than the tens digit, and the thousands digit is 1 less than the ones digit. As a consequence, the ones digit of the result, which is the difference of the digits of the ANNA number, determines the whole result.

All operative proofs are rigorous as they rest on *generally applicable operations* independently of the examples by which they are demonstrated.

This learning environment is well suited to illustrate three general points:

(1) Teaching aids must be selected carefully, and the decisive criterion is how well they incorporate the mathematical structure and can be used for operative proofs (Wittmann 1998).

(2) Decisions about which methods of calculating are preferable can only be made by looking *at the whole curriculum*. Otherwise, not only will the consistency and the coherence be adversely affected but so will the mathematical impact.

(3) In guiding students through a learning environment, the teacher has mainly to follow the natural flow of the mathematical activity rooted essentially in "well-understood mathematics" as it is perfectly captured in Guy Brousseau's theory of didactical situations: introduction, action, communication, validation, institutionalization (Brousseau 1997).

3.4 *Addressing Teachers as "Reflective Practitioners"*

The new handbook stimulates readers to become active not only mathematically but also didactically. In this regard, the most important new feature in the handbook is an adaptation of the Japanese lesson study method (see Becker and Shimada 1997; Hirabayashi 2002). Each chapter of the handbook ends with some proposals for conducting "collective teaching experiments." This term was inspired by the French philosopher Bruno Latour, who introduced it in environmental sociology (for details see Wittmann 2016).

Teachers must be made aware that for systemic reasons, researchers cannot collect and communicate all knowledge that is needed for teaching. So teachers must sensibly implement what is proposed to them, adapt it to their experiences and routines, and collect further information in the classroom themselves.[7]

The following is a typical example for a collective teaching experiment: Readers are invited to compare the proposal for guiding students through the learning environment "ANNA numbers" with another approach in which a similar environment about "UHU Numbers" is first explored.[8] It is interesting to observe to which extent students are able to transfer what they have learned about UHU numbers to ANNA numbers.

All collective teaching experiments in the handbook are numbered to facilitate an exchange about students' experiences.

4 Final Remarks

1. "Well-understood mathematics" should not only be integrated into didactical courses, textbooks and materials for teachers. To organize *mathematical courses* in teacher education accordingly would greatly contribute to improving the image of mathematics, both with teachers and student teachers, and to provide them with highly effective professional knowledge. This knowledge could be further developed in didactical courses ("methods courses").
   It seems promising to design mathematical courses for primary teachers starting with the mathematics that is integrated in the handbook. To rephrase Dewey: In this way, student teachers would "constantly get object-lessons of the finest type in the kind of mental activity which characterizes mental growth and, hence, the educative process."

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[7] Hiro Ninomiya has sensibly pointed to the importance of Japanese teachers' "implicit" knowledge for conducting lessons. The present author has often found that the systemic thinking for which he has to plea fervently in his context is implicit in Japanese education in many ways—so implicit that in Japanese there is not even a word for "systemic.".

[8] UHU numbers are numbers like 343, 727, etc. "Uhu" is the German name for eagle owl. For any two pairs of different digits, there are two UHU numbers which can be subtracted: 434 − 343, 727 − 272, ... The results 91, 182, 273, ..., 819 also show a conspicuous pattern and are multiples of 91.

2. Although structure-genetic didactical analyses provide a great deal of empirical evidence for teaching, it is worthwhile to conduct controlled teaching experiments and to document the processes that can be observed. Documentation of this kind is very valuable in teacher education (Hirabayashi 2002). Here, young researchers will find ample opportunities.

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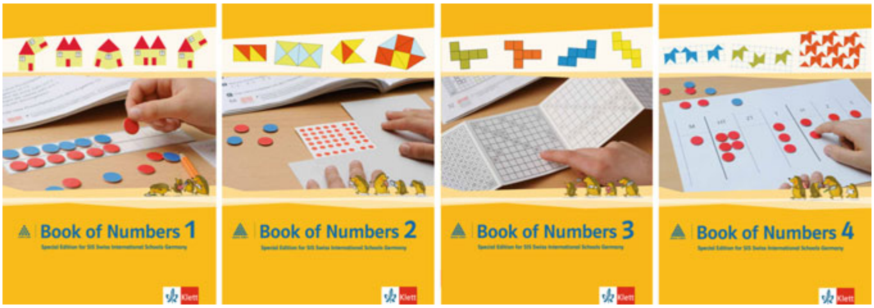
Appendix
Excerpts from *The Book of Numbers* (BN)

> *This: Combining thinking and doing*
> *This: Inducing students to combine thinking and doing*
> *is the source point of any productive education.*
>
> *Friedrich Froebel 1821*

The following pages should provide some impression of how learning environments developed in Mathe 2000 have been implemented in a textbook. The e-Book version of this volume shows these pages in color.

The pages have also been selected with the intention to convey an idea of the coherent and consistent mathematical structure of the textbook *The Book of Numbers* which is a truly mathematical book. In line with the quotation from page 1, which is repeated above, the readers are invited to solve the problems offered on these pages themselves. This will give them a real impression of the conception of the *Book of Numbers* in which the choice and the sequence of problems are well-considered.



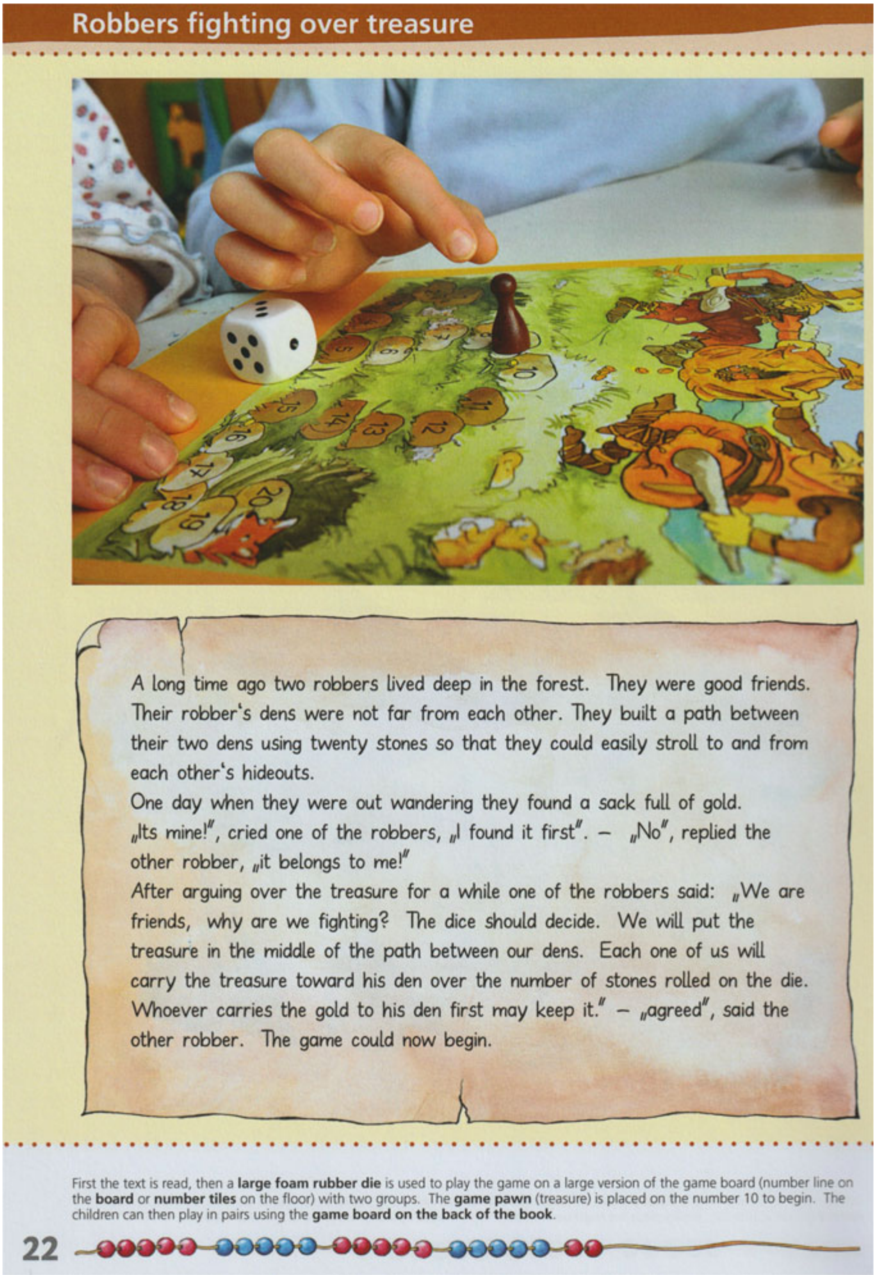
E. C. Wittmann, *Connecting Mathematics and Mathematics Education*,
https://doi.org/10.1007/978-3-030-61570-3

Pages	Content	Source		Reference
p. 290	Game for developing the knowledge of the number line, and first experiences with random experiments	BN Early Math. vol. 2, p. 22		Chapter 3
p. 287	Investigation and operative proof of the well-known relationship between triangular and square numbers	BN 2,	p. 116	Chapter 5
pp. 292–293	Introduction of arithmogons in grade 1, using counters that allow for experimental solutions	BN 1,	pp. 74–75	Chapter 6
pp. 294–295	Continuing work with arithmogons, using counters for solving the difficult case	BN 1,	pp. 106–107	
p. 296	Arithmogons: Systematic solution of the difficult case	BN 3,	p. 120	Chapter 6
pp. 297–300	Introduction of the addition table in grade 1. The commutative and the associative law of addition are used for deriving the results of more difficult problems from easy ones. Counters and the Twenty frame allow for making relationships explicit.	BN 1,	pp. 52–55	Chapter 8
pp. 301–305	Introduction of the multiplication table in grade 2. The commutative and associative law of multiplication and the distributive law are used for deriving the results of more difficult problems from easy ones. Hundred array and angle card allow for making relationships explicit.	BN 2,	pp. 67–71	Chapter 13
pp. 306–309	Holistic introduction of the thousand space in grade 3. Teaching aids: Thousand book and Thousand array	BN 3,	pp. 32–35	Chapter 8
p. 310	Investigation of a number pattern within the multiplication table, operative proof of the pattern with dot arrays	BN 2,	p. 117	Chapter 9

Pages	Content	Source		Reference
pp. 311–312	Investigation and operative proof of patterns with arrow strings in grade 3	BN 3,	p. 3, p. 113	Chapter 11
p. 313	The strategy "Tens minus Tens, Ones minus Ones" introduced in grade 2 (see the solution of Tim, exercise 3, at the bottom)	BN 2,	p. 50	Chapter 12
p. 314	The strategy "Tens minus Tens, Ones minus Ones" continued in grade 3 (see the solution of Mika)	BN 3,	p. 68	
p. 315	Tessellations with polyominoes as a preparation of the nets of a cube	BN 3,	p. 11	Chapter 13
pp. 316–317	Operative proof of the rule "Casting out Nines" with the place value chart and counters	BN 3,	p. 122–123	Chapter 13
p. 318	Investigation of the patterns with "ANNA numbers"	BN 4,	p. 118	Chapter 14

The textbook pages BN 1, p. 55, BN 2, 71, BN 3, pp. 33, 35, contain references to the course *Calculightning* (s. Chap. 14). This course is available also in the form of Apps with an option in English. There are 5 Apps: *Calculightning 0* for Early Math, *Calculightning 1 – 4* for the vols. 1 – 4 of the *Book of Numbers*. These Apps can be found in the App Store or at Google Play by entering "Klett Calculightning".

Readers who are interested in learning more about the project Mathe 2000 and about the materials developed in this project are referred to the website www.Mathe2000.de that contains a section in English.



Robbers fighting over treasure

A long time ago two robbers lived deep in the forest. They were good friends. Their robber‘s dens were not far from each other. They built a path between their two dens using twenty stones so that they could easily stroll to and from each other‘s hideouts.

One day when they were out wandering they found a sack full of gold.

„Its mine!", cried one of the robbers, „I found it first". – „No", replied the other robber, „it belongs to me!"

After arguing over the treasure for a while one of the robbers said: „We are friends, why are we fighting? The dice should decide. We will put the treasure in the middle of the path between our dens. Each one of us will carry the treasure toward his den over the number of stones rolled on the die. Whoever carries the gold to his den first may keep it." – „agreed", said the other robber. The game could now begin.

First the text is read, then a **large foam rubber die** is used to play the game on a large version of the game board (number line on the **board** or **number tiles** on the floor) with two groups. The **game pawn** (treasure) is placed on the number 10 to begin. The children can then play in pairs using the **game board on the back of the book**.

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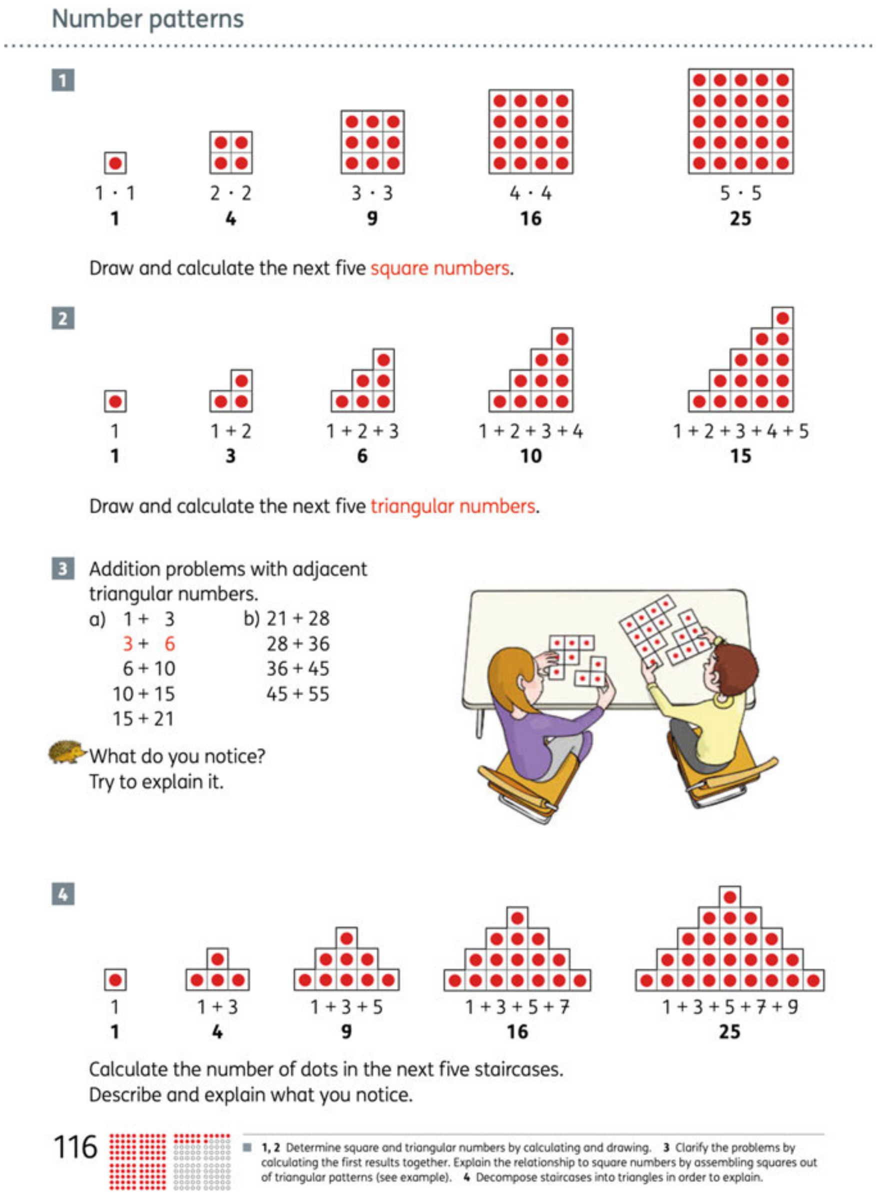


Number patterns

1

1 · 1	2 · 2	3 · 3	4 · 4	5 · 5
1	**4**	**9**	**16**	**25**

Draw and calculate the next five square numbers.

2

1	1 + 2	1 + 2 + 3	1 + 2 + 3 + 4	1 + 2 + 3 + 4 + 5
1	**3**	**6**	**10**	**15**

Draw and calculate the next five triangular numbers.

3 Addition problems with adjacent triangular numbers.

a) 1 + 3
3 + 6
6 + 10
10 + 15
15 + 21

b) 21 + 28
28 + 36
36 + 45
45 + 55

What do you notice?
Try to explain it.

4

1	1 + 3	1 + 3 + 5	1 + 3 + 5 + 7	1 + 3 + 5 + 7 + 9
1	**4**	**9**	**16**	**25**

Calculate the number of dots in the next five staircases.
Describe and explain what you notice.

116

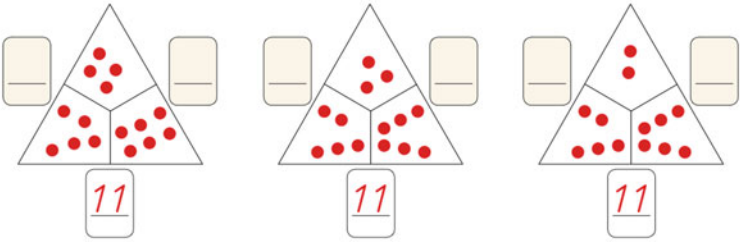
1, 2 Determine square and triangular numbers by calculating and drawing. **3** Clarify the problems by calculating the first results together. Explain the relationship to square numbers by assembling squares out of triangular patterns (see example). **4** Decompose staircases into triangles in order to explain.

Arithmogons

1 How do you do arithmogons? Explain.

Calculate, compare, describe.

2



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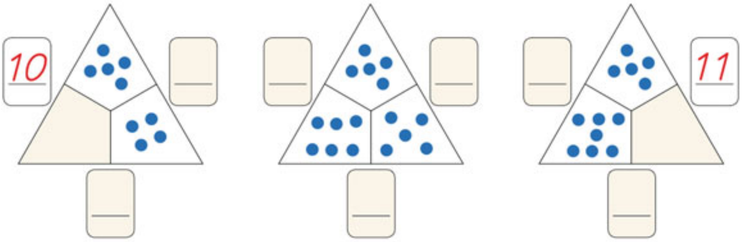
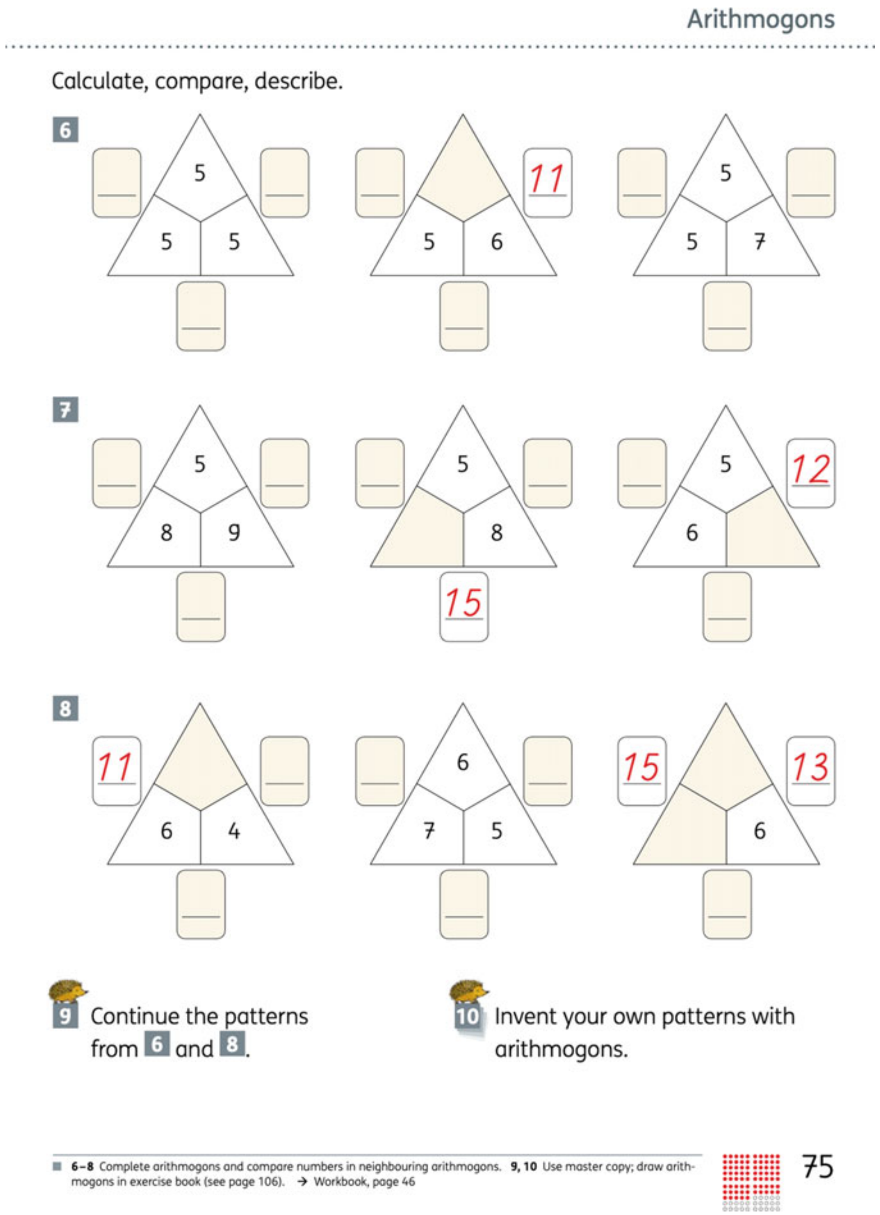


4 Continue the patterns from **2** and **3**.

5 Think of your own arithmogons. Calculate them.

1 Explain the structure of arithmogons (numbers on the **inside left**, **inside right**, **inside top**, **outside left**, **outside right**, **outside bottom**), solve several problems together. Rule: Write down the sum of neighbouring inside numbers on the outside. **2, 3** Calculate missing numbers using a large arithmogon and counters. **4** Describe and continue the patterns of problems and results in **2** and **3**. **5** If desired, draw arithmogons in exercise book (see page 106).



Arithmogons

Calculate, compare, describe.

6

7

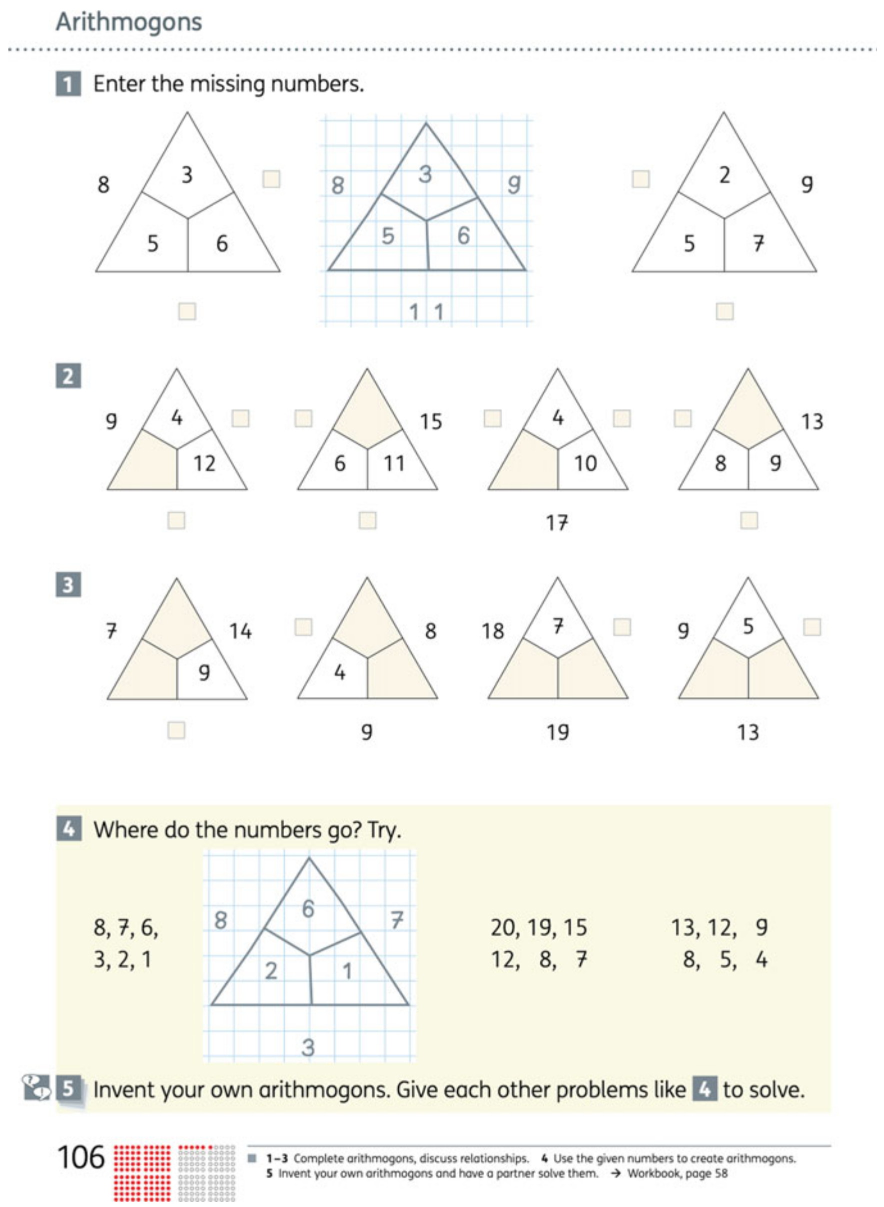
8

9 Continue the patterns from **6** and **8**.

10 Invent your own patterns with arithmogons.

6–8 Complete arithmogons and compare numbers in neighbouring arithmogons. **9, 10** Use master copy; draw arithmogons in exercise book (see page 106). → Workbook, page 46

75



Arithmogons

1 Enter the missing numbers.

2

3

4 Where do the numbers go? Try.

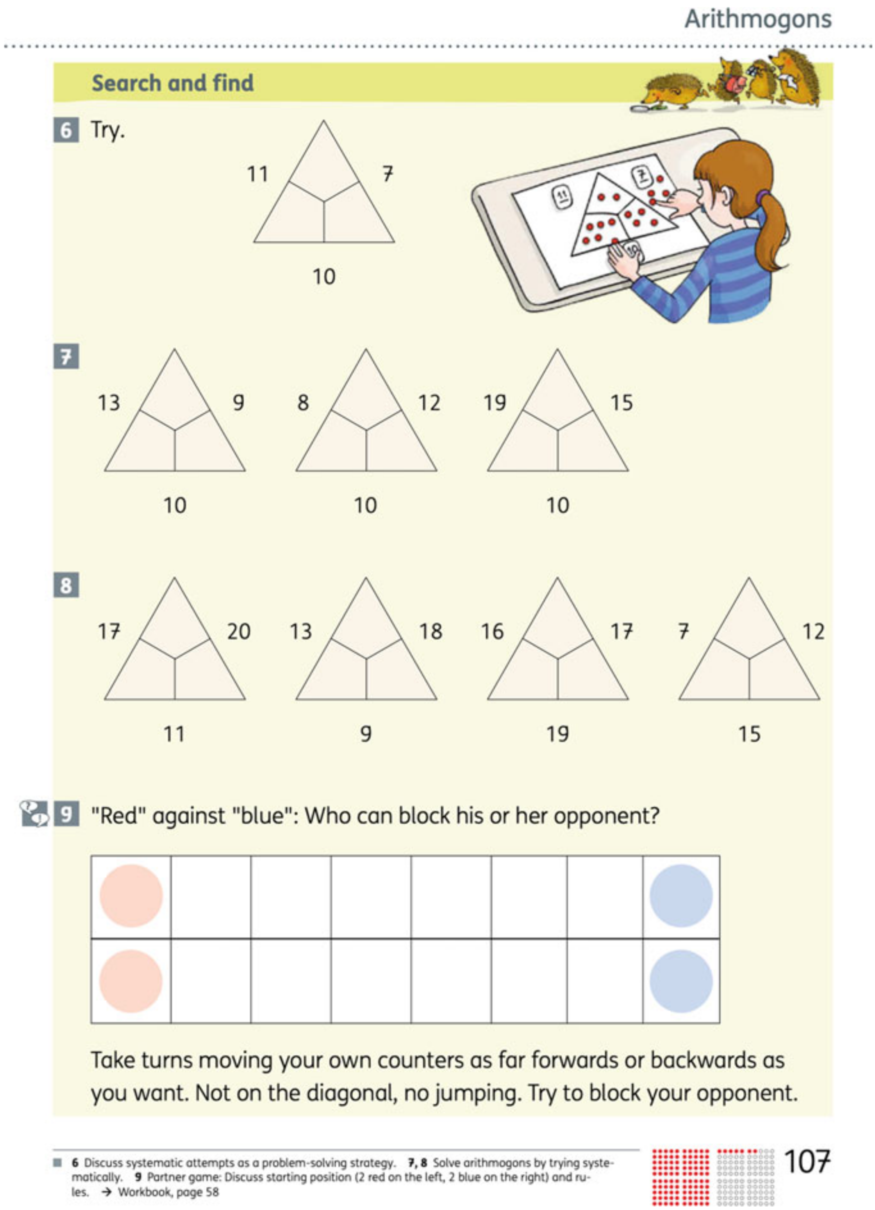
8, 7, 6,
3, 2, 1

20, 19, 15
12, 8, 7

13, 12, 9
8, 5, 4

5 Invent your own arithmogons. Give each other problems like **4** to solve.

1–3 Complete arithmogons, discuss relationships. **4** Use the given numbers to create arithmogons.
5 Invent your own arithmogons and have a partner solve them. → Workbook, page 58



Arithmogons

Search and find

6 Try.

11 7
10

7

13 9
10

8 12
10

19 15
10

8

17 20
11

13 18
9

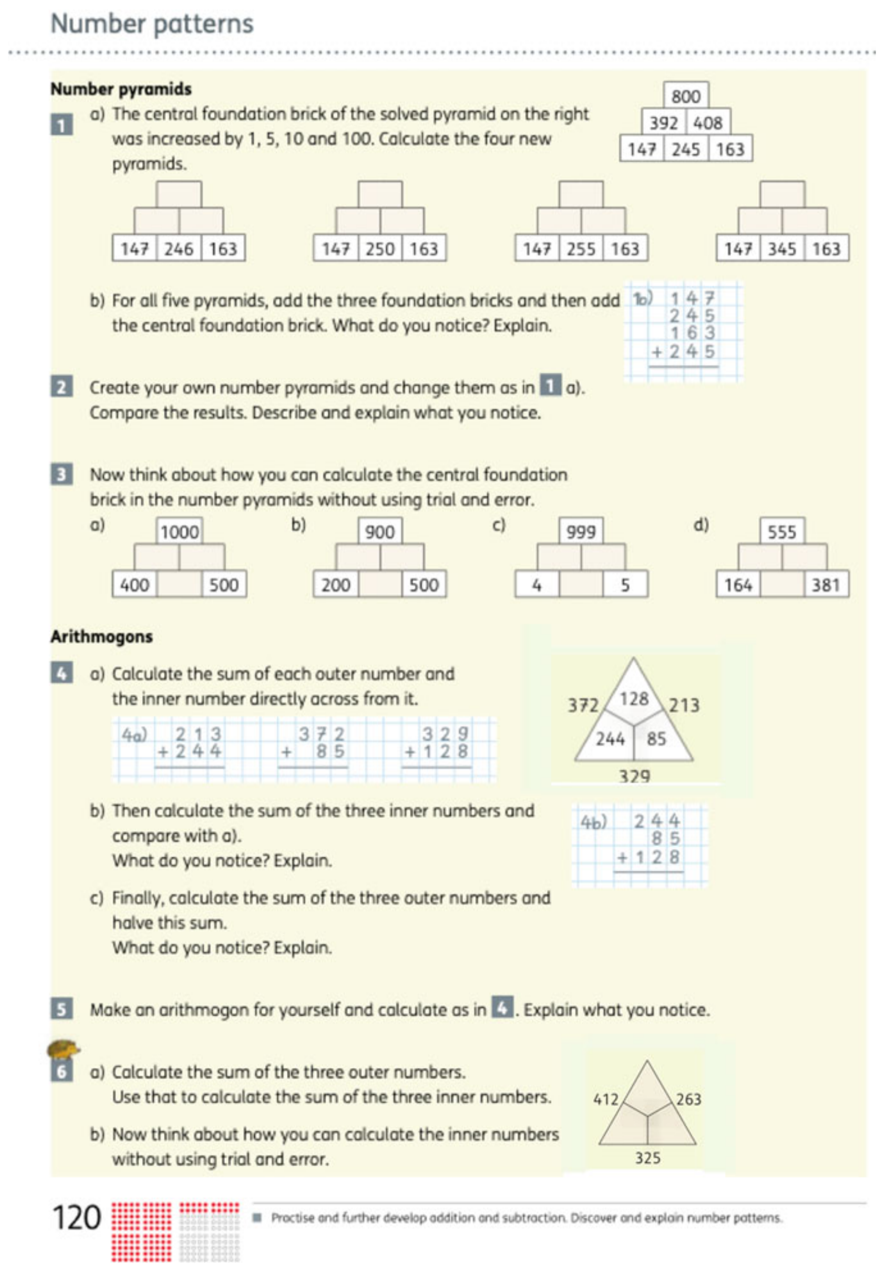
16 17
19

7 12
15

9 "Red" against "blue": Who can block his or her opponent?

Take turns moving your own counters as far forwards or backwards as you want. Not on the diagonal, no jumping. Try to block your opponent.

■ **6** Discuss systematic attempts as a problem-solving strategy. **7, 8** Solve arithmogons by trying systematically. **9** Partner game: Discuss starting position (2 red on the left, 2 blue on the right) and rules. → Workbook, page 58



Number patterns

Number pyramids

1 a) The central foundation brick of the solved pyramid on the right was increased by 1, 5, 10 and 100. Calculate the four new pyramids.

800
392 | 408
147 | 245 | 163

147 | 246 | 163

147 | 250 | 163

147 | 255 | 163

147 | 345 | 163

b) For all five pyramids, add the three foundation bricks and then add the central foundation brick. What do you notice? Explain.

1b) 147
245
163
+ 245

2 Create your own number pyramids and change them as in **1** a). Compare the results. Describe and explain what you notice.

3 Now think about how you can calculate the central foundation brick in the number pyramids without using trial and error.

a) 1000
400 | | 500

b) 900
200 | | 500

c) 999
4 | | 5

d) 555
164 | | 381

Arithmogons

4 a) Calculate the sum of each outer number and the inner number directly across from it.

4a) 213
+ 244

372
+ 85

329
+ 128

372, 128, 213, 244, 85, 329

b) Then calculate the sum of the three inner numbers and compare with a). What do you notice? Explain.

4b) 244
85
+ 128

c) Finally, calculate the sum of the three outer numbers and halve this sum. What do you notice? Explain.

5 Make an arithmogon for yourself and calculate as in **4**. Explain what you notice.

6 a) Calculate the sum of the three outer numbers. Use that to calculate the sum of the three inner numbers.

b) Now think about how you can calculate the inner numbers without using trial and error.

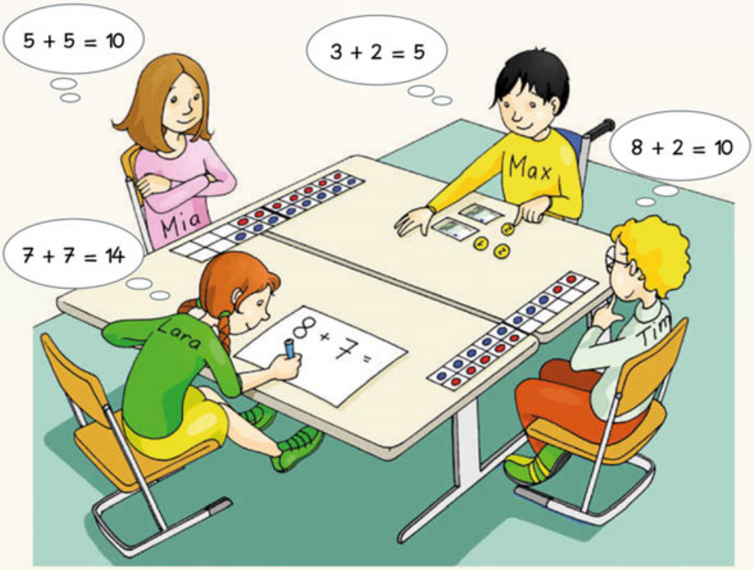
412, 263, 325

120

■ Practise and further develop addition and subtraction. Discover and explain number patterns.

Ways to calculate

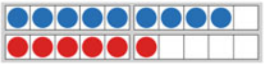
1 How do you calculate 8 + 7?



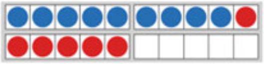
How do the children calculate? Which easy problems do they use?
Compare with how you calculated.

2 How do the children calculate the problem 9 + 6 = 15?

Mia:



Tim:



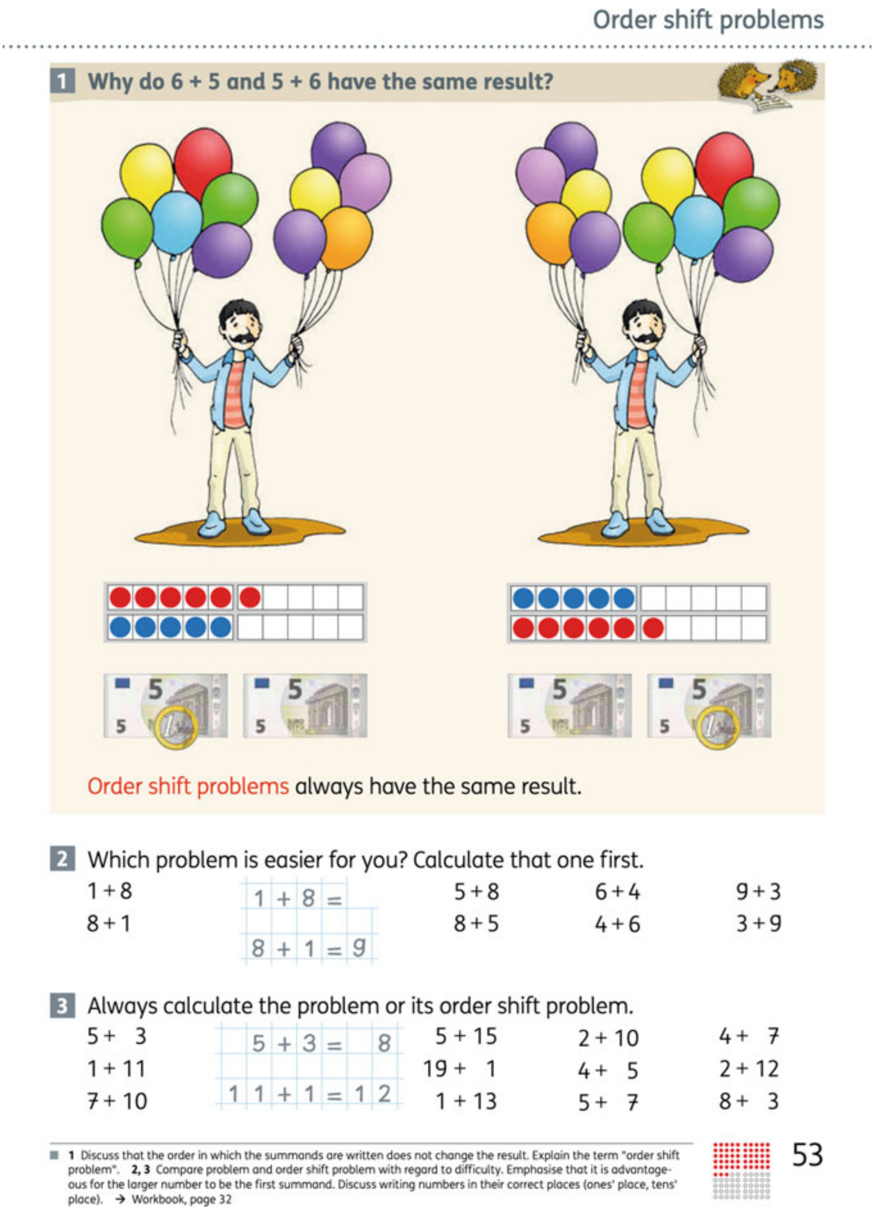
Max:

Lara calculates: 10 + 5 = 15

3 How do you calculate?

9 + 7 6 + 9 8 + 5 8 + 6 5 + 7

■ **1** Have the children calculate the problem 8 + 7 first. Discuss how they calculated (maths conference). Discuss how Mia, Lara, Tim and Max calculated. Point out the easy problems that were used (problems with 5, doubling, complementing to 10). Explain to children that they may choose their own ways of calculating. **2** Calculate problem. Discuss how they solved the problems together. **3** Encourage students to solve problems in their own way.



Order shift problems

1 **Why do 6 + 5 and 5 + 6 have the same result?**

Order shift problems always have the same result.

2 Which problem is easier for you? Calculate that one first.

1 + 8
8 + 1

1 + 8 =
8 + 1 = 9

5 + 8
8 + 5

6 + 4
4 + 6

9 + 3
3 + 9

3 Always calculate the problem or its order shift problem.

5 + 3
1 + 11
7 + 10

5 + 3 = 8
11 + 1 = 12

5 + 15
19 + 1
1 + 13

2 + 10
4 + 5
5 + 7

4 + 7
2 + 12
8 + 3

■ **1** Discuss that the order in which the summands are written does not change the result. Explain the term "order shift problem". **2, 3** Compare problem and order shift problem with regard to difficulty. Emphasise that it is advantageous for the larger number to be the first summand. Discuss writing numbers in their correct places (ones' place, tens' place). → Workbook, page 32

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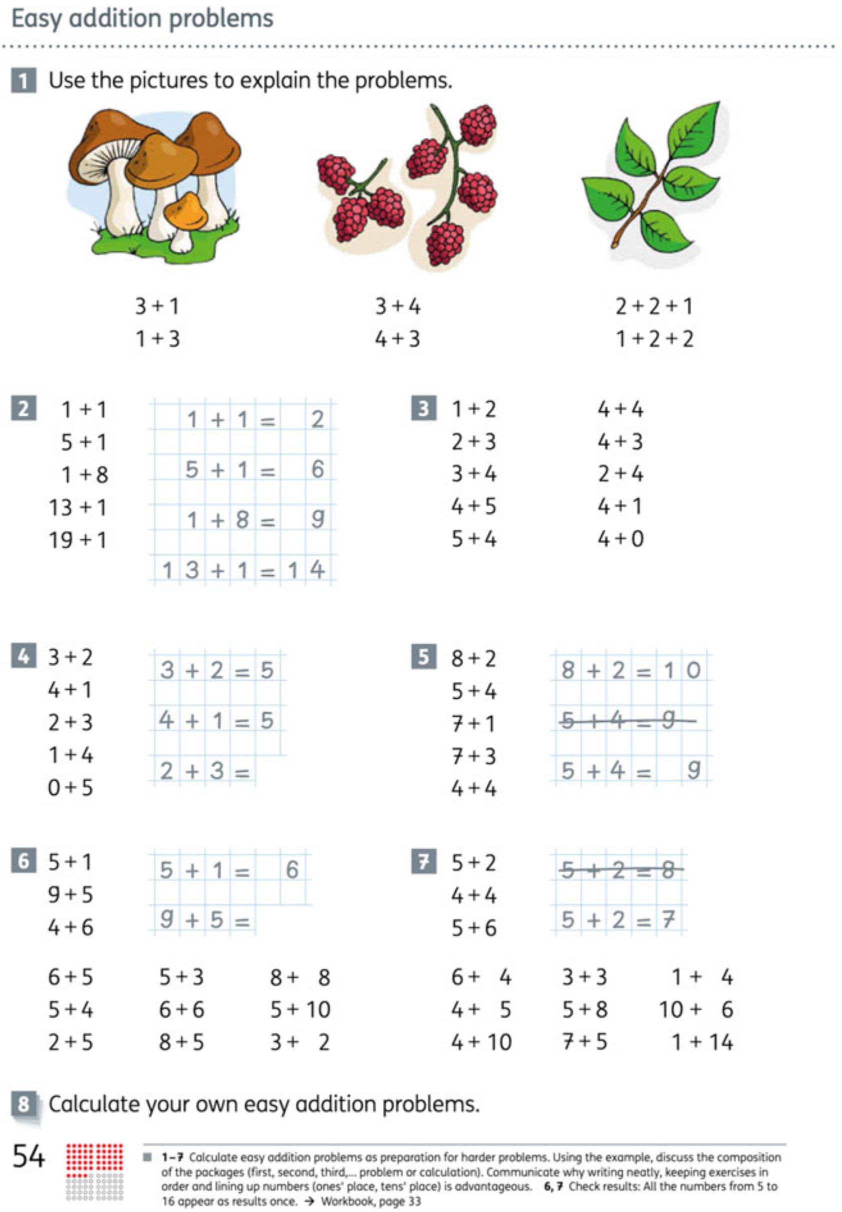


Easy addition problems

1 Use the pictures to explain the problems.

3 + 1
1 + 3

3 + 4
4 + 3

2 + 2 + 1
1 + 2 + 2

2
1 + 1
5 + 1
1 + 8
13 + 1
19 + 1

1 + 1 = 2
5 + 1 = 6
1 + 8 = 9
13 + 1 = 14

3
1 + 2
2 + 3
3 + 4
4 + 5
5 + 4

4 + 4
4 + 3
2 + 4
4 + 1
4 + 0

4
3 + 2
4 + 1
2 + 3
1 + 4
0 + 5

3 + 2 = 5
4 + 1 = 5
2 + 3 =

5
8 + 2
5 + 4
7 + 1
7 + 3
4 + 4

8 + 2 = 10
~~5 + 4 = 9~~
5 + 4 = 9

6
5 + 1
9 + 5
4 + 6

5 + 1 = 6
9 + 5 =

6 + 5
5 + 4
2 + 5

5 + 3
6 + 6
8 + 5

8 + 8
5 + 10
3 + 2

7
5 + 2
4 + 4
5 + 6

~~5 + 2 = 8~~
5 + 2 = 7

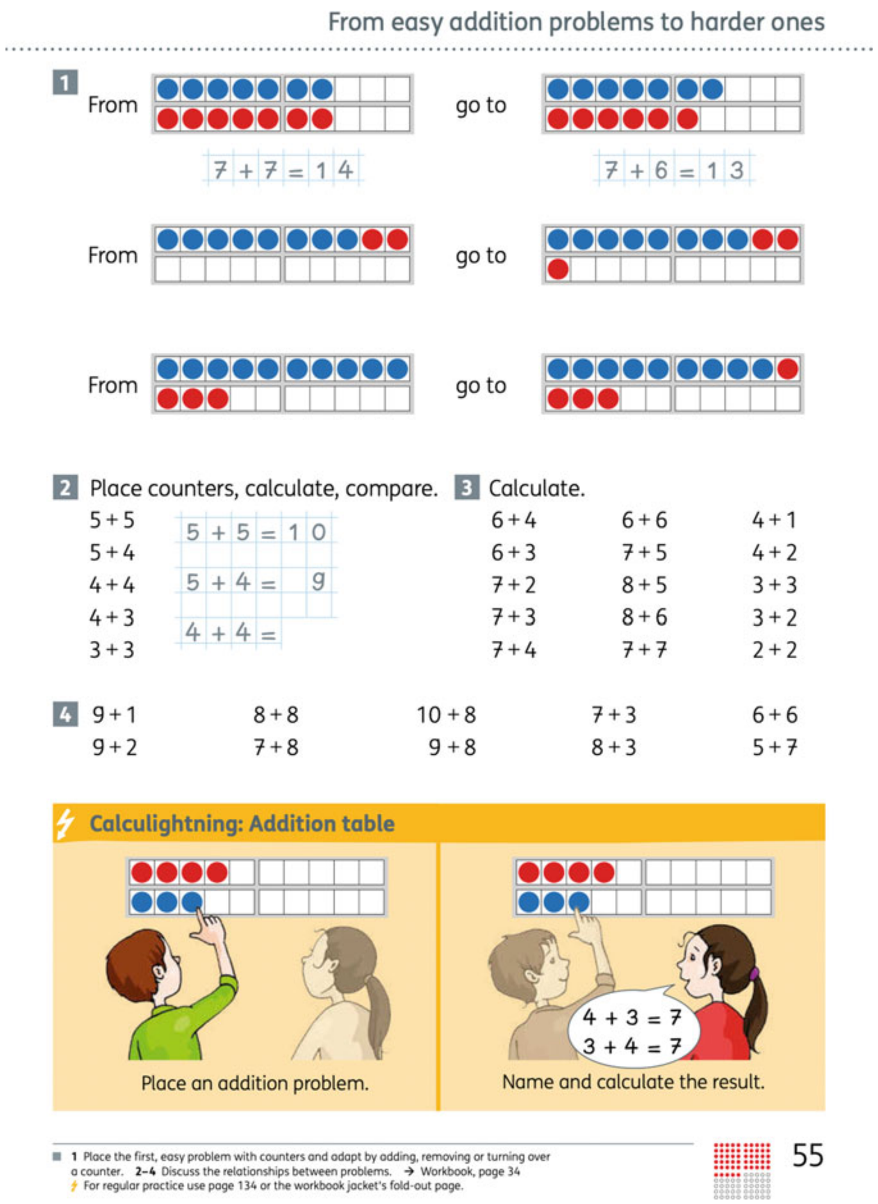
6 + 4
4 + 5
4 + 10

3 + 3
5 + 8
7 + 5

1 + 4
10 + 6
1 + 14

8 Calculate your own easy addition problems.

■ **1–7** Calculate easy addition problems as preparation for harder problems. Using the example, discuss the composition of the packages (first, second, third,... problem or calculation). Communicate why writing neatly, keeping exercises in order and lining up numbers (ones' place, tens' place) is advantageous. **6, 7** Check results: All the numbers from 5 to 16 appear as results once. → Workbook, page 33



From easy addition problems to harder ones

1 From … go to …

7 + 7 = 14

7 + 6 = 13

From … go to …

From … go to …

2 Place counters, calculate, compare.

5 + 5
5 + 4
4 + 4
4 + 3
3 + 3

5 + 5 = 10
5 + 4 = 9
4 + 4 =

3 Calculate.

6 + 4
6 + 3
7 + 2
7 + 3
7 + 4

6 + 6
7 + 5
8 + 5
8 + 6
7 + 7

4 + 1
4 + 2
3 + 3
3 + 2
2 + 2

4

9 + 1
9 + 2

8 + 8
7 + 8

10 + 8
9 + 8

7 + 3
8 + 3

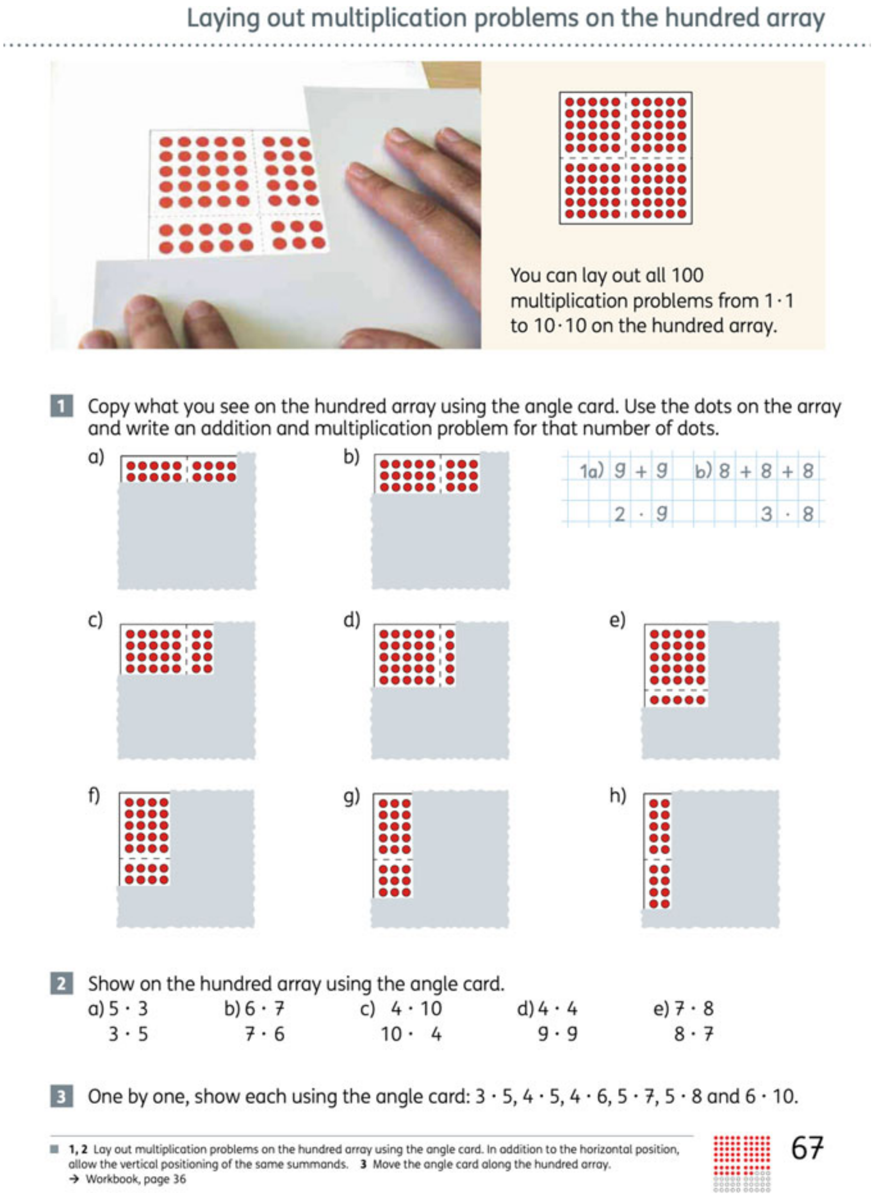
6 + 6
5 + 7

Calculightning: Addition table

Place an addition problem.

Name and calculate the result.

1 Place the first, easy problem with counters and adapt by adding, removing or turning over a counter. **2–4** Discuss the relationships between problems. → Workbook, page 34
For regular practice use page 134 or the workbook jacket's fold-out page.



Laying out multiplication problems on the hundred array

You can lay out all 100 multiplication problems from 1 · 1 to 10 · 10 on the hundred array.

1 Copy what you see on the hundred array using the angle card. Use the dots on the array and write an addition and multiplication problem for that number of dots.

1a) 9 + 9 b) 8 + 8 + 8
2 · 9 3 · 8

2 Show on the hundred array using the angle card.

a) 5 · 3 b) 6 · 7 c) 4 · 10 d) 4 · 4 e) 7 · 8
3 · 5 7 · 6 10 · 4 9 · 9 8 · 7

3 One by one, show each using the angle card: 3 · 5, 4 · 5, 4 · 6, 5 · 7, 5 · 8 and 6 · 10.

■ **1, 2** Lay out multiplication problems on the hundred array using the angle card. In addition to the horizontal position, allow the vertical positioning of the same summands. **3** Move the angle card along the hundred array.
→ Workbook, page 36

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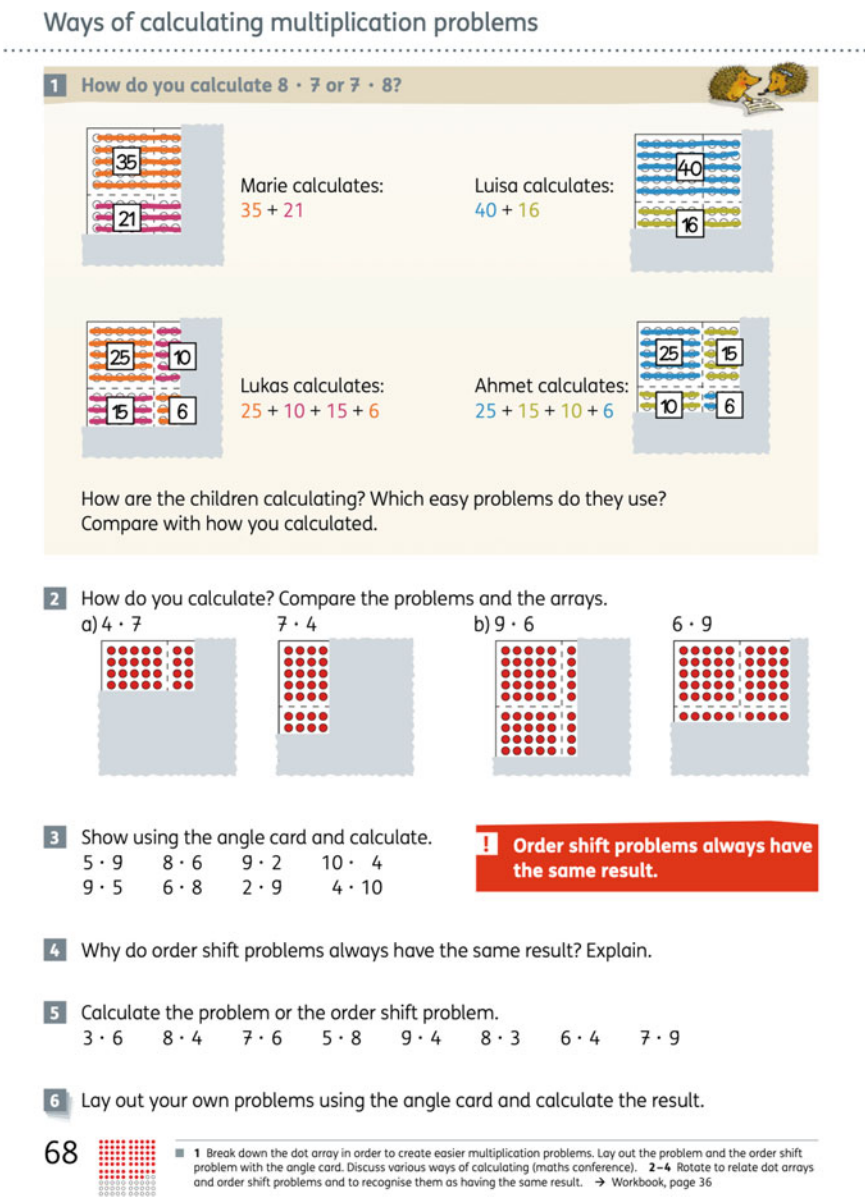

Ways of calculating multiplication problems

1 **How do you calculate 8 · 7 or 7 · 8?**

Marie calculates:
35 + 21

Luisa calculates:
40 + 16

Lukas calculates:
25 + 10 + 15 + 6

Ahmet calculates:
25 + 15 + 10 + 6

How are the children calculating? Which easy problems do they use?
Compare with how you calculated.

2 How do you calculate? Compare the problems and the arrays.
a) 4 · 7 7 · 4 b) 9 · 6 6 · 9

3 Show using the angle card and calculate.
5 · 9 8 · 6 9 · 2 10 · 4
9 · 5 6 · 8 2 · 9 4 · 10

! Order shift problems always have the same result.

4 Why do order shift problems always have the same result? Explain.

5 Calculate the problem or the order shift problem.
3 · 6 8 · 4 7 · 6 5 · 8 9 · 4 8 · 3 6 · 4 7 · 9

6 Lay out your own problems using the angle card and calculate the result.

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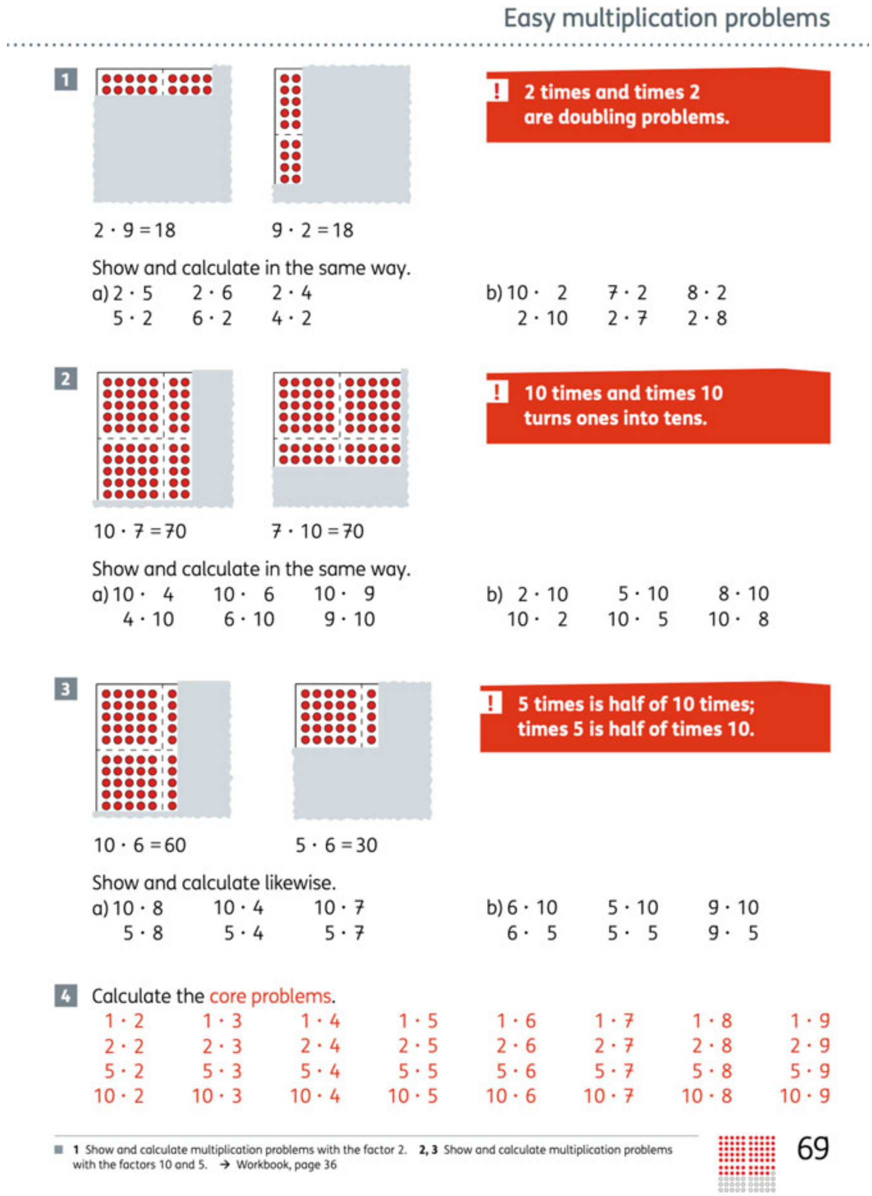
■ **1** Break down the dot array in order to create easier multiplication problems. Lay out the problem and the order shift problem with the angle card. Discuss various ways of calculating (maths conference). **2–4** Rotate to relate dot arrays and order shift problems and to recognise them as having the same result. → Workbook, page 36



Easy multiplication problems

1

$2 \cdot 9 = 18$ $9 \cdot 2 = 18$

! 2 times and times 2 are doubling problems.

Show and calculate in the same way.

a) $2 \cdot 5$ $2 \cdot 6$ $2 \cdot 4$
$5 \cdot 2$ $6 \cdot 2$ $4 \cdot 2$

b) $10 \cdot 2$ $7 \cdot 2$ $8 \cdot 2$
$2 \cdot 10$ $2 \cdot 7$ $2 \cdot 8$

2

$10 \cdot 7 = 70$ $7 \cdot 10 = 70$

! 10 times and times 10 turns ones into tens.

Show and calculate in the same way.

a) $10 \cdot 4$ $10 \cdot 6$ $10 \cdot 9$
$4 \cdot 10$ $6 \cdot 10$ $9 \cdot 10$

b) $2 \cdot 10$ $5 \cdot 10$ $8 \cdot 10$
$10 \cdot 2$ $10 \cdot 5$ $10 \cdot 8$

3

$10 \cdot 6 = 60$ $5 \cdot 6 = 30$

! 5 times is half of 10 times; times 5 is half of times 10.

Show and calculate likewise.

a) $10 \cdot 8$ $10 \cdot 4$ $10 \cdot 7$
$5 \cdot 8$ $5 \cdot 4$ $5 \cdot 7$

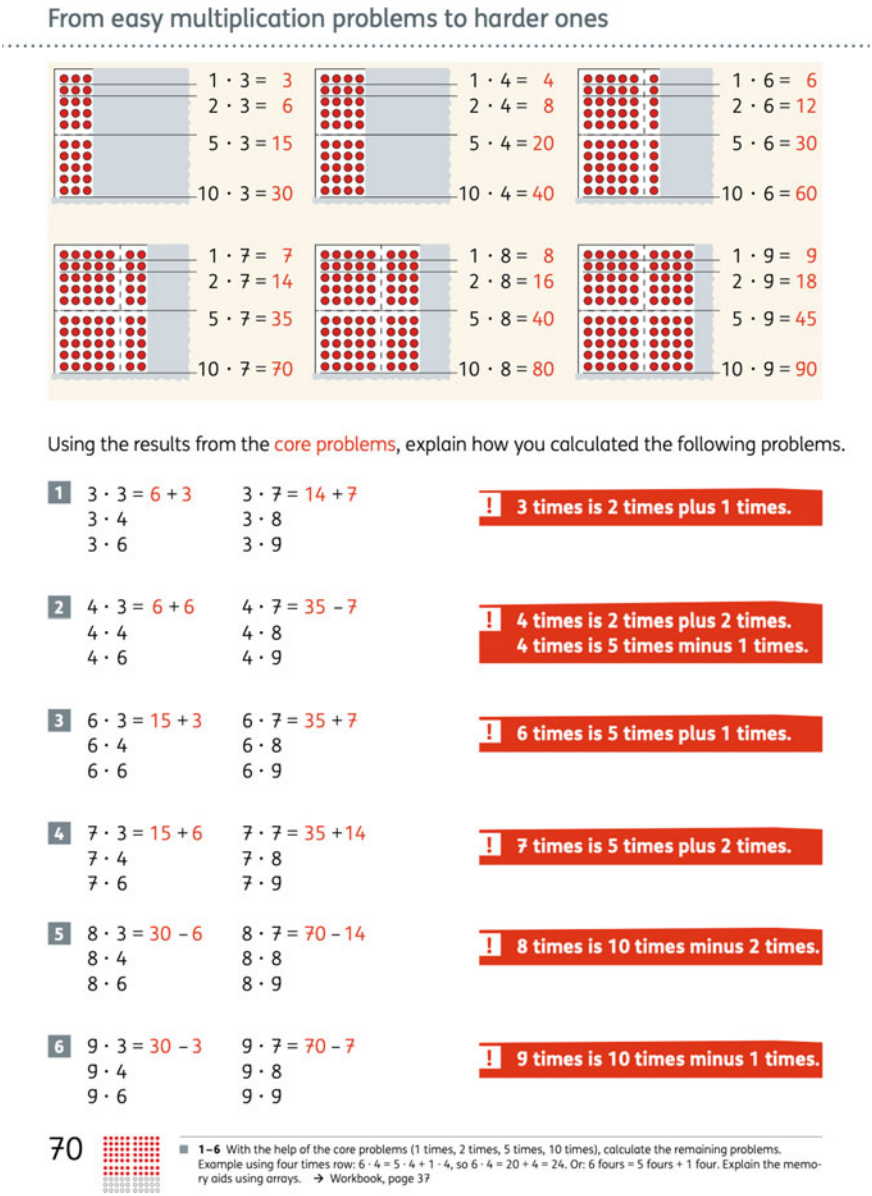
b) $6 \cdot 10$ $5 \cdot 10$ $9 \cdot 10$
$6 \cdot 5$ $5 \cdot 5$ $9 \cdot 5$

4 Calculate the core problems.

$1 \cdot 2$ $1 \cdot 3$ $1 \cdot 4$ $1 \cdot 5$ $1 \cdot 6$ $1 \cdot 7$ $1 \cdot 8$ $1 \cdot 9$
$2 \cdot 2$ $2 \cdot 3$ $2 \cdot 4$ $2 \cdot 5$ $2 \cdot 6$ $2 \cdot 7$ $2 \cdot 8$ $2 \cdot 9$
$5 \cdot 2$ $5 \cdot 3$ $5 \cdot 4$ $5 \cdot 5$ $5 \cdot 6$ $5 \cdot 7$ $5 \cdot 8$ $5 \cdot 9$
$10 \cdot 2$ $10 \cdot 3$ $10 \cdot 4$ $10 \cdot 5$ $10 \cdot 6$ $10 \cdot 7$ $10 \cdot 8$ $10 \cdot 9$

■ **1** Show and calculate multiplication problems with the factor 2. **2, 3** Show and calculate multiplication problems with the factors 10 and 5. → Workbook, page 36

69



From easy multiplication problems to harder ones

$1 \cdot 3 = 3$
$2 \cdot 3 = 6$
$5 \cdot 3 = 15$
$10 \cdot 3 = 30$

$1 \cdot 4 = 4$
$2 \cdot 4 = 8$
$5 \cdot 4 = 20$
$10 \cdot 4 = 40$

$1 \cdot 6 = 6$
$2 \cdot 6 = 12$
$5 \cdot 6 = 30$
$10 \cdot 6 = 60$

$1 \cdot 7 = 7$
$2 \cdot 7 = 14$
$5 \cdot 7 = 35$
$10 \cdot 7 = 70$

$1 \cdot 8 = 8$
$2 \cdot 8 = 16$
$5 \cdot 8 = 40$
$10 \cdot 8 = 80$

$1 \cdot 9 = 9$
$2 \cdot 9 = 18$
$5 \cdot 9 = 45$
$10 \cdot 9 = 90$

Using the results from the core problems, explain how you calculated the following problems.

1 $3 \cdot 3 = 6 + 3$ $\quad 3 \cdot 7 = 14 + 7$
$3 \cdot 4$ $\quad 3 \cdot 8$
$3 \cdot 6$ $\quad 3 \cdot 9$

! 3 times is 2 times plus 1 times.

2 $4 \cdot 3 = 6 + 6$ $\quad 4 \cdot 7 = 35 - 7$
$4 \cdot 4$ $\quad 4 \cdot 8$
$4 \cdot 6$ $\quad 4 \cdot 9$

! 4 times is 2 times plus 2 times.
4 times is 5 times minus 1 times.

3 $6 \cdot 3 = 15 + 3$ $\quad 6 \cdot 7 = 35 + 7$
$6 \cdot 4$ $\quad 6 \cdot 8$
$6 \cdot 6$ $\quad 6 \cdot 9$

! 6 times is 5 times plus 1 times.

4 $7 \cdot 3 = 15 + 6$ $\quad 7 \cdot 7 = 35 + 14$
$7 \cdot 4$ $\quad 7 \cdot 8$
$7 \cdot 6$ $\quad 7 \cdot 9$

! 7 times is 5 times plus 2 times.

5 $8 \cdot 3 = 30 - 6$ $\quad 8 \cdot 7 = 70 - 14$
$8 \cdot 4$ $\quad 8 \cdot 8$
$8 \cdot 6$ $\quad 8 \cdot 9$

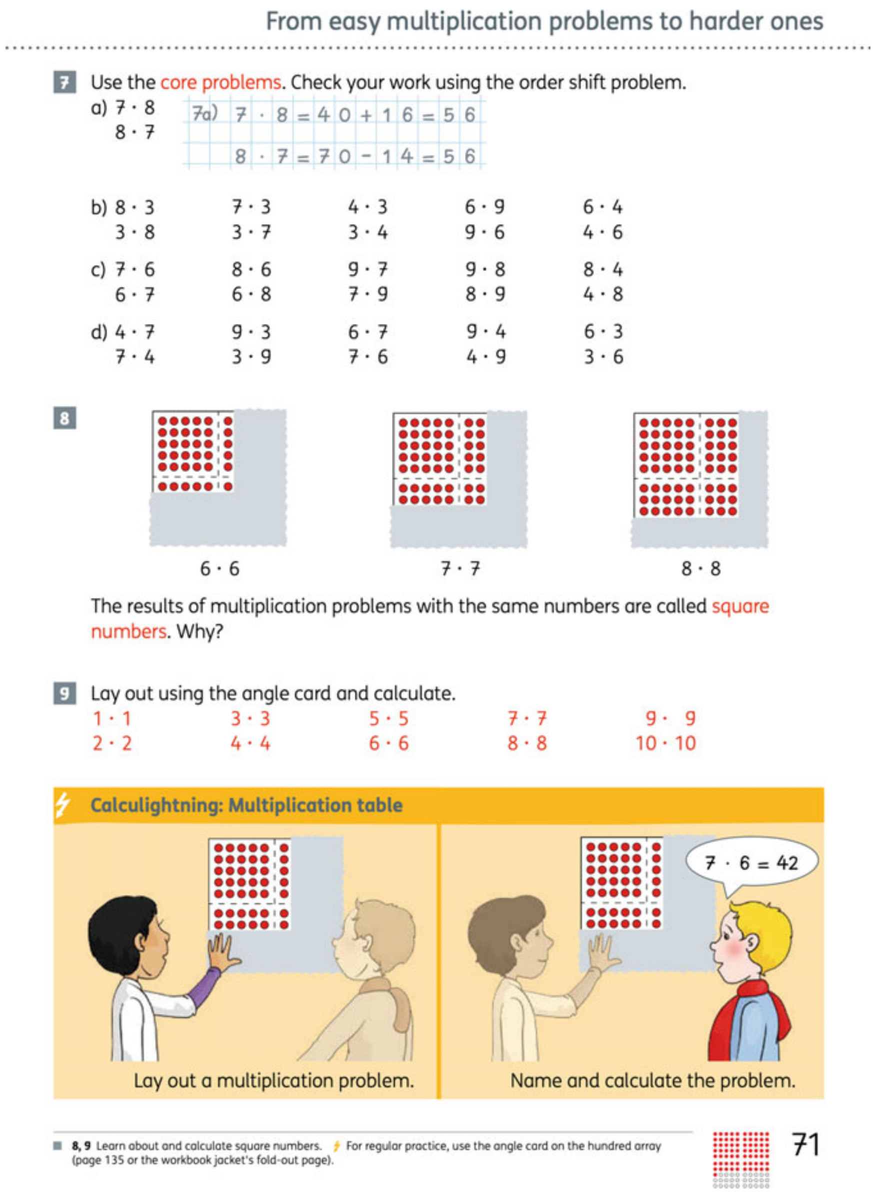
! 8 times is 10 times minus 2 times.

6 $9 \cdot 3 = 30 - 3$ $\quad 9 \cdot 7 = 70 - 7$
$9 \cdot 4$ $\quad 9 \cdot 8$
$9 \cdot 6$ $\quad 9 \cdot 9$

! 9 times is 10 times minus 1 times.

70

1–6 With the help of the core problems (1 times, 2 times, 5 times, 10 times), calculate the remaining problems. Example using four times row: $6 \cdot 4 = 5 \cdot 4 + 1 \cdot 4$, so $6 \cdot 4 = 20 + 4 = 24$. Or: 6 fours = 5 fours + 1 four. Explain the memory aids using arrays. → Workbook, page 37



From easy multiplication problems to harder ones

7 Use the core problems. Check your work using the order shift problem.

a) 7 · 8
8 · 7

7a) 7 · 8 = 40 + 16 = 56
8 · 7 = 70 − 14 = 56

b) 8 · 3, 3 · 8 — 7 · 3, 3 · 7 — 4 · 3, 3 · 4 — 6 · 9, 9 · 6 — 6 · 4, 4 · 6

c) 7 · 6, 6 · 7 — 8 · 6, 6 · 8 — 9 · 7, 7 · 9 — 9 · 8, 8 · 9 — 8 · 4, 4 · 8

d) 4 · 7, 7 · 4 — 9 · 3, 3 · 9 — 6 · 7, 7 · 6 — 9 · 4, 4 · 9 — 6 · 3, 3 · 6

8

6 · 6 7 · 7 8 · 8

The results of multiplication problems with the same numbers are called square numbers. Why?

9 Lay out using the angle card and calculate.

1 · 1, 2 · 2 — 3 · 3, 4 · 4 — 5 · 5, 6 · 6 — 7 · 7, 8 · 8 — 9 · 9, 10 · 10

Calculightning: Multiplication table

Lay out a multiplication problem.

Name and calculate the problem.

8, 9 Learn about and calculate square numbers. For regular practice, use the angle card on the hundred array (page 135 or the workbook jacket's fold-out page).

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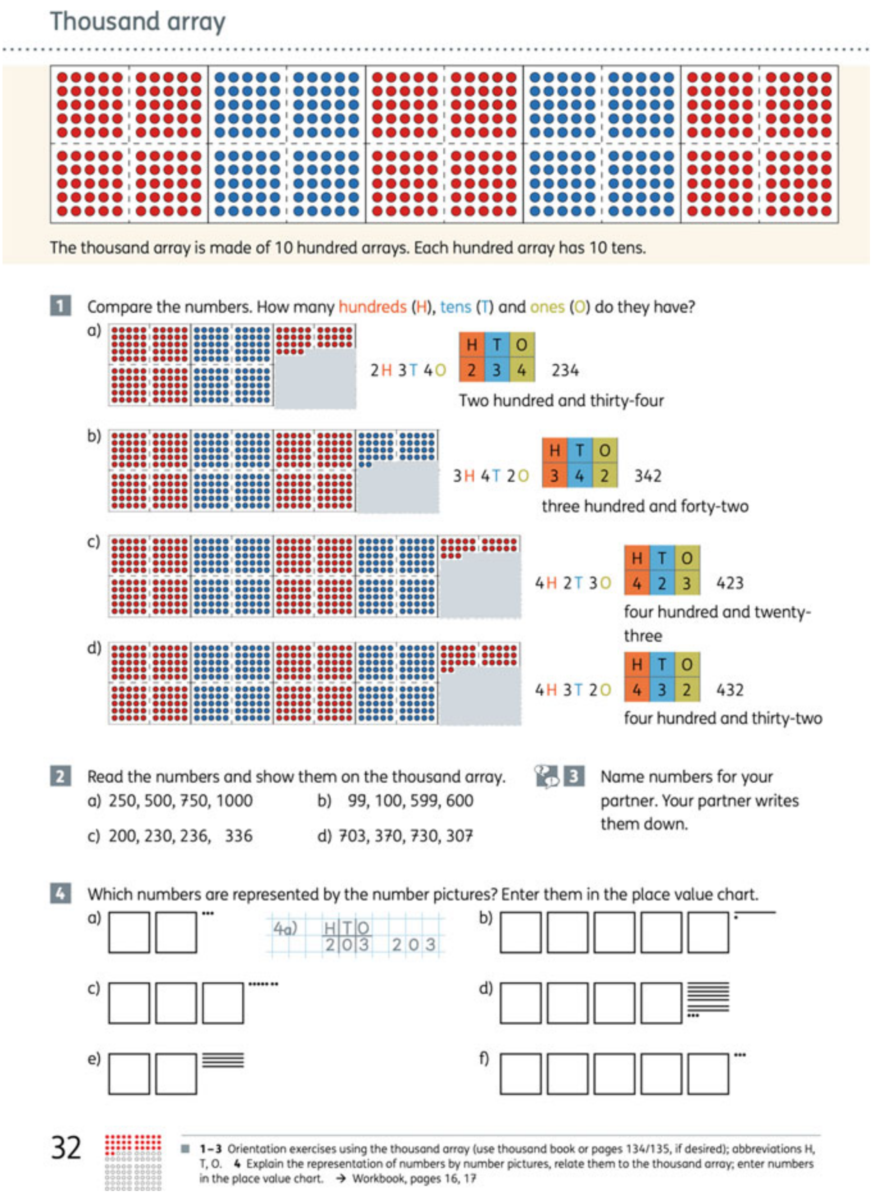


Thousand array

The thousand array is made of 10 hundred arrays. Each hundred array has 10 tens.

1 Compare the numbers. How many hundreds (H), tens (T) and ones (O) do they have?

a) 2H 3T 4O

H	T	O
2	3	4

234

Two hundred and thirty-four

b) 3H 4T 2O

H	T	O
3	4	2

342

three hundred and forty-two

c) 4H 2T 3O

H	T	O
4	2	3

423

four hundred and twenty-three

d) 4H 3T 2O

H	T	O
4	3	2

432

four hundred and thirty-two

2 Read the numbers and show them on the thousand array.

a) 250, 500, 750, 1000
b) 99, 100, 599, 600
c) 200, 230, 236, 336
d) 703, 370, 730, 307

3 Name numbers for your partner. Your partner writes them down.

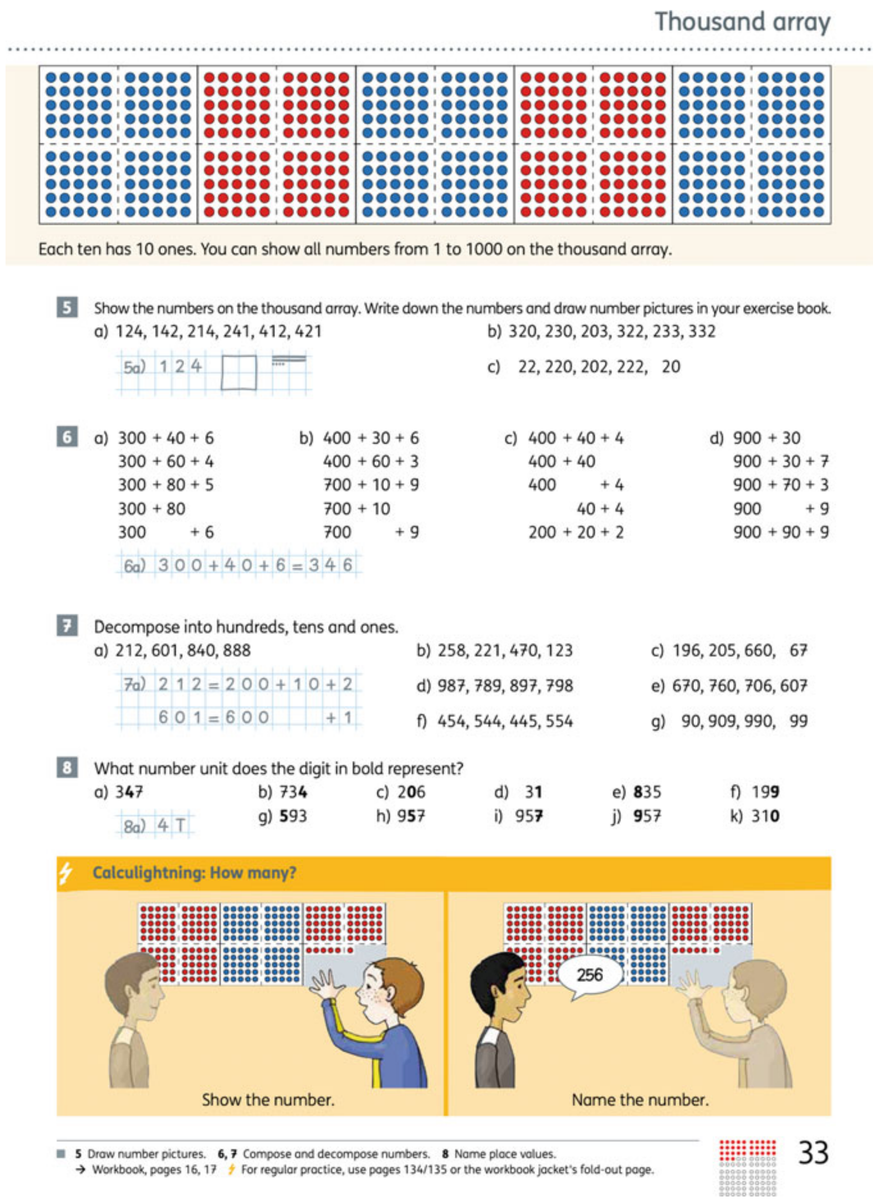
4 Which numbers are represented by the number pictures? Enter them in the place value chart.

a) b) c) d) e) f)

4a) H T O
2 0 3 203

32

1–3 Orientation exercises using the thousand array (use thousand book or pages 134/135, if desired); abbreviations H, T, O. 4 Explain the representation of numbers by number pictures, relate them to the thousand array; enter numbers in the place value chart. → Workbook, pages 16, 17



Thousand array

Each ten has 10 ones. You can show all numbers from 1 to 1000 on the thousand array.

5 Show the numbers on the thousand array. Write down the numbers and draw number pictures in your exercise book.

a) 124, 142, 214, 241, 412, 421

5a) 1 2 4

b) 320, 230, 203, 322, 233, 332

c) 22, 220, 202, 222, 20

6

a) 300 + 40 + 6
300 + 60 + 4
300 + 80 + 5
300 + 80
300 + 6

b) 400 + 30 + 6
400 + 60 + 3
700 + 10 + 9
700 + 10
700 + 9

c) 400 + 40 + 4
400 + 40
400 + 4
40 + 4
200 + 20 + 2

d) 900 + 30
900 + 30 + 7
900 + 70 + 3
900 + 9
900 + 90 + 9

6a) 3 0 0 + 4 0 + 6 = 3 4 6

7 Decompose into hundreds, tens and ones.

a) 212, 601, 840, 888

7a) 2 1 2 = 2 0 0 + 1 0 + 2
6 0 1 = 6 0 0 + 1

b) 258, 221, 470, 123

c) 196, 205, 660, 67

d) 987, 789, 897, 798

e) 670, 760, 706, 607

f) 454, 544, 445, 554

g) 90, 909, 990, 99

8 What number unit does the digit in bold represent?

a) 3**4**7
b) 73**4**
c) 2**0**6
d) 3**1**
e) **8**35
f) 19**9**
g) **5**93
h) 9**5**7
i) 95**7**
j) **9**57
k) 31**0**

8a) 4 T

Calculightning: How many?

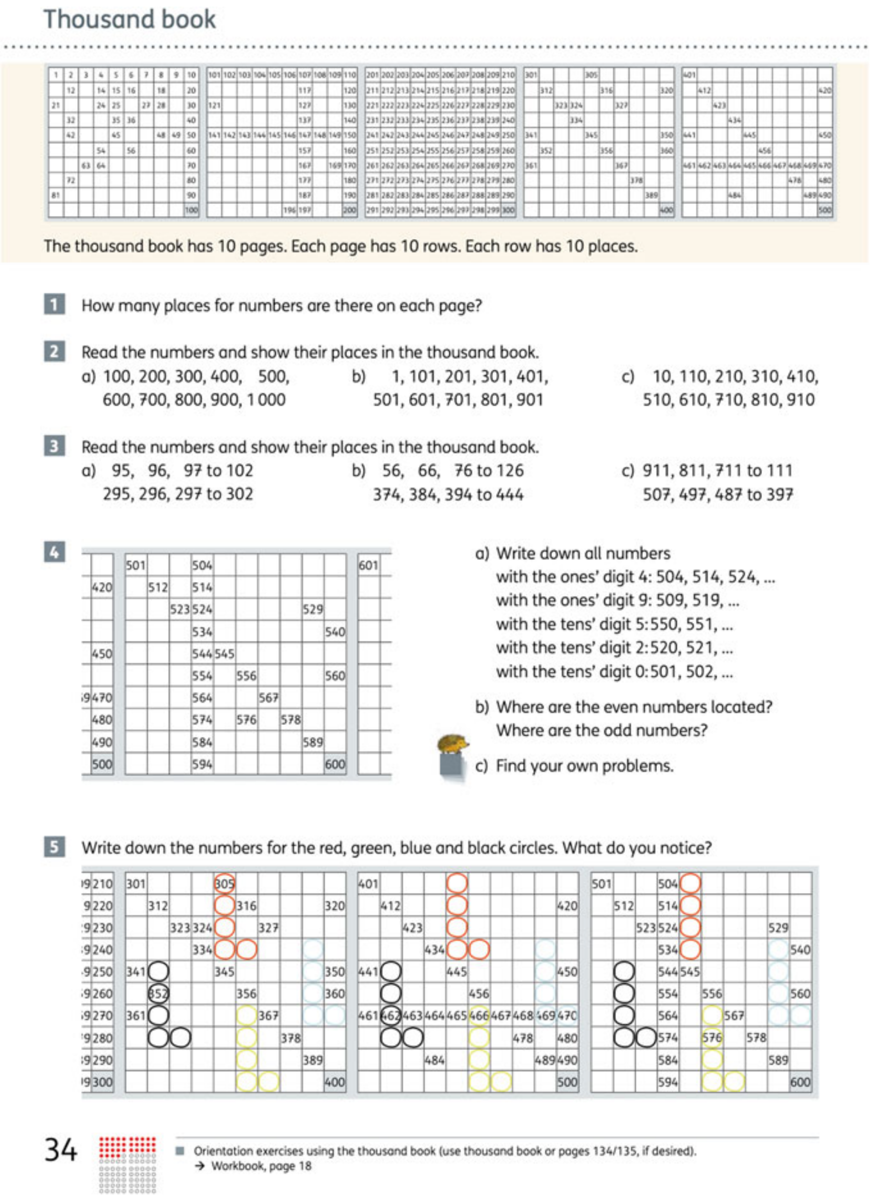
Show the number.

256

Name the number.

■ **5** Draw number pictures. **6, 7** Compose and decompose numbers. **8** Name place values.
→ Workbook, pages 16, 17 For regular practice, use pages 134/135 or the workbook jacket's fold-out page.

33



Thousand book

The thousand book has 10 pages. Each page has 10 rows. Each row has 10 places.

1 How many places for numbers are there on each page?

2 Read the numbers and show their places in the thousand book.

a) 100, 200, 300, 400, 500, 600, 700, 800, 900, 1 000

b) 1, 101, 201, 301, 401, 501, 601, 701, 801, 901

c) 10, 110, 210, 310, 410, 510, 610, 710, 810, 910

3 Read the numbers and show their places in the thousand book.

a) 95, 96, 97 to 102
295, 296, 297 to 302

b) 56, 66, 76 to 126
374, 384, 394 to 444

c) 911, 811, 711 to 111
507, 497, 487 to 397

4

	501			504							601
420		512		514							
			523	524					529		
				534						540	
450				544	545						
				554		556				560	
470				564			567				
480				574		576		578			
490				584					589		
500				594						600	

a) Write down all numbers
with the ones' digit 4: 504, 514, 524, ...
with the ones' digit 9: 509, 519, ...
with the tens' digit 5: 550, 551, ...
with the tens' digit 2: 520, 521, ...
with the tens' digit 0: 501, 502, ...

b) Where are the even numbers located?
Where are the odd numbers?

c) Find your own problems.

5 Write down the numbers for the red, green, blue and black circles. What do you notice?

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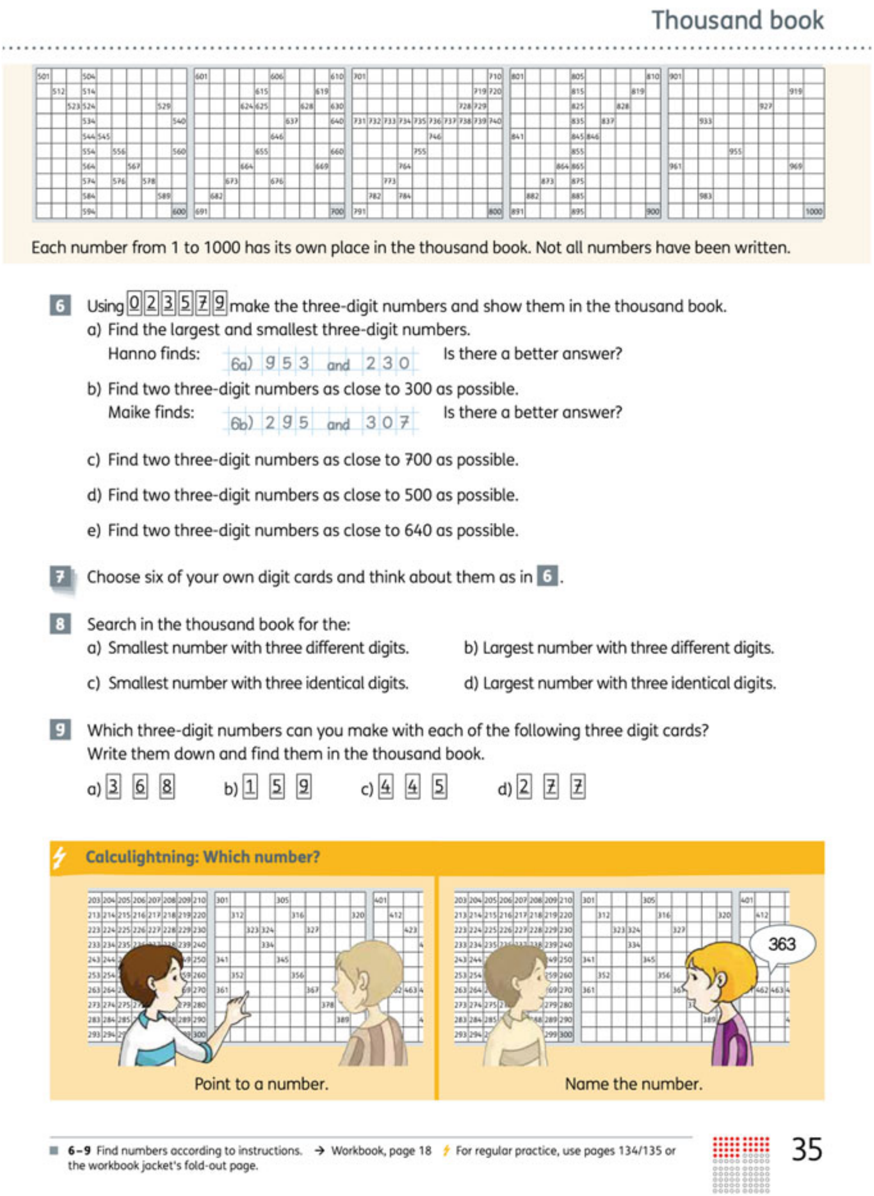
- Orientation exercises using the thousand book (use thousand book or pages 134/135, if desired).
→ Workbook, page 18



Thousand book

501			504						
	512		514						
		523	524					529	
			534						540
			544	545					
			554		556				560
			564			567			
			574		576		578		
			584					589	
			594						600

601					606				610
				615				619	
			624	625			628		630
						637			640
					646				
				655					660
			664					669	
		673			676				
	682								
691									700

701									710
								719	720
							728	729	
731	732	733	734	735	736	737	738	739	740
					746				
				755					
			764						
		773							
	782		784						
791									800

801				805					810
				815				819	
				825			828		
				835		837			
841				845	846				
				855					
			864	865					
		873		875					
	882			885					
891				895					900

901									
								919	
						927			
		933							
				955					
961								969	
		983							
									1000

Each number from 1 to 1000 has its own place in the thousand book. Not all numbers have been written.

6 Using 0 2 3 5 7 9 make the three-digit numbers and show them in the thousand book.

a) Find the largest and smallest three-digit numbers.

Hanno finds: 6a) 9 5 3 and 2 3 0 Is there a better answer?

b) Find two three-digit numbers as close to 300 as possible.

Maike finds: 6b) 2 9 5 and 3 0 7 Is there a better answer?

c) Find two three-digit numbers as close to 700 as possible.

d) Find two three-digit numbers as close to 500 as possible.

e) Find two three-digit numbers as close to 640 as possible.

7 Choose six of your own digit cards and think about them as in **6**.

8 Search in the thousand book for the:

a) Smallest number with three different digits.
b) Largest number with three different digits.
c) Smallest number with three identical digits.
d) Largest number with three identical digits.

9 Which three-digit numbers can you make with each of the following three digit cards? Write them down and find them in the thousand book.

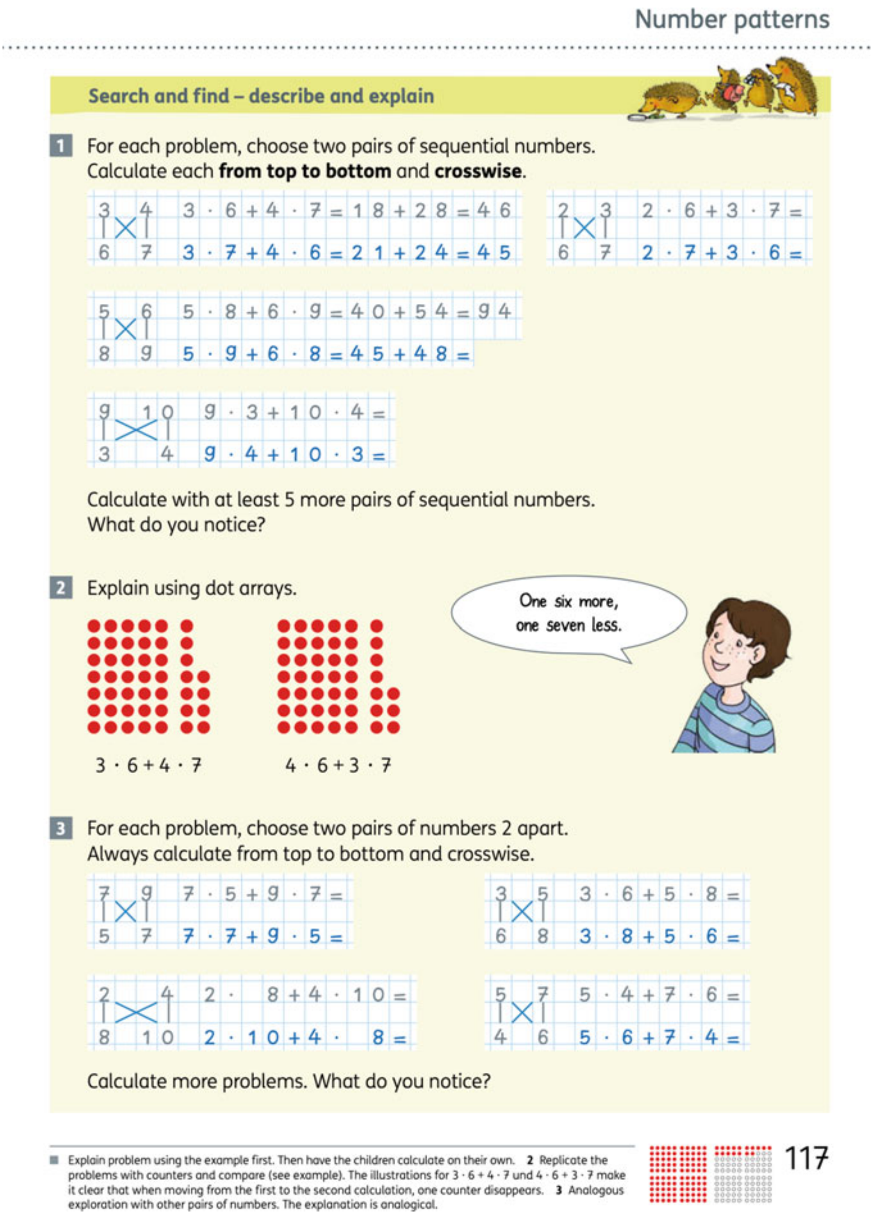
a) 3 6 8 b) 1 5 9 c) 4 4 5 d) 2 7 7

Calculightning: Which number?

Point to a number.

Name the number.

■ **6–9** Find numbers according to instructions. → Workbook, page 18 ⚡ For regular practice, use pages 134/135 or the workbook jacket's fold-out page.



Number patterns

Search and find – describe and explain

1 For each problem, choose two pairs of sequential numbers.
Calculate each **from top to bottom** and **crosswise**.

3 4 3 · 6 + 4 · 7 = 18 + 28 = 46
6 7 3 · 7 + 4 · 6 = 21 + 24 = 45

2 3 2 · 6 + 3 · 7 =
6 7 2 · 7 + 3 · 6 =

5 6 5 · 8 + 6 · 9 = 40 + 54 = 94
8 9 5 · 9 + 6 · 8 = 45 + 48 =

9 10 9 · 3 + 10 · 4 =
3 4 9 · 4 + 10 · 3 =

Calculate with at least 5 more pairs of sequential numbers.
What do you notice?

2 Explain using dot arrays.

3 · 6 + 4 · 7

4 · 6 + 3 · 7

3 For each problem, choose two pairs of numbers 2 apart.
Always calculate from top to bottom and crosswise.

7 9 7 · 5 + 9 · 7 =
5 7 7 · 7 + 9 · 5 =

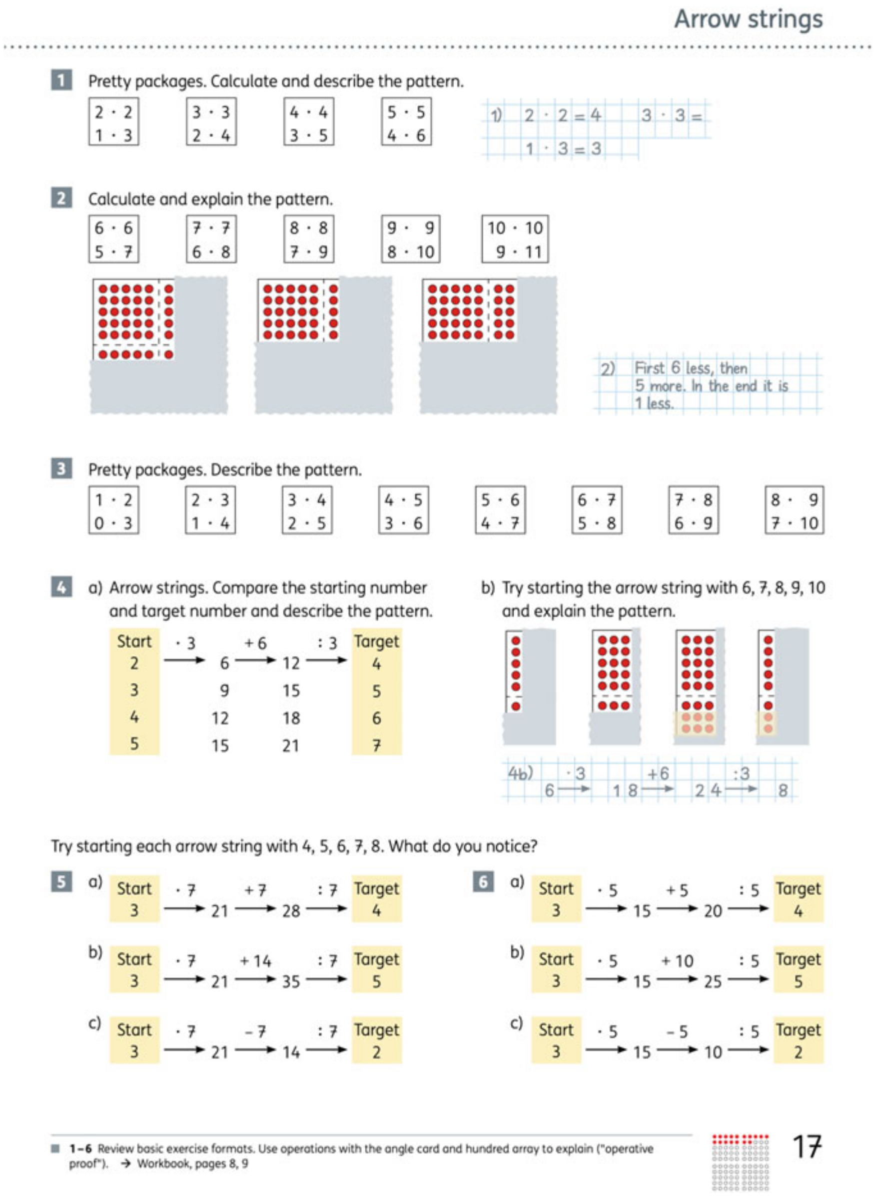
3 5 3 · 6 + 5 · 8 =
6 8 3 · 8 + 5 · 6 =

2 4 2 · 8 + 4 · 10 =
8 10 2 · 10 + 4 · 8 =

5 7 5 · 4 + 7 · 6 =
4 6 5 · 6 + 7 · 4 =

Calculate more problems. What do you notice?

■ Explain problem using the example first. Then have the children calculate on their own. **2** Replicate the problems with counters and compare (see example). The illustrations for 3 · 6 + 4 · 7 und 4 · 6 + 3 · 7 make it clear that when moving from the first to the second calculation, one counter disappears. **3** Analogous exploration with other pairs of numbers. The explanation is analogical.



Arrow strings

1 Pretty packages. Calculate and describe the pattern.

2 · 2 1 · 3	3 · 3 2 · 4	4 · 4 3 · 5	5 · 5 4 · 6

1) 2 · 2 = 4   3 · 3 =
1 · 3 = 3

2 Calculate and explain the pattern.

6 · 6 5 · 7	7 · 7 6 · 8	8 · 8 7 · 9	9 · 9 8 · 10	10 · 10 9 · 11

2) First 6 less, then
5 more. In the end it is
1 less.

3 Pretty packages. Describe the pattern.

1 · 2 0 · 3	2 · 3 1 · 4	3 · 4 2 · 5	4 · 5 3 · 6	5 · 6 4 · 7	6 · 7 5 · 8	7 · 8 6 · 9	8 · 9 7 · 10

4 a) Arrow strings. Compare the starting number and target number and describe the pattern.

Start	· 3		+ 6		: 3	Target
2	→	6	→	12	→	4
3		9		15		5
4		12		18		6
5		15		21		7

b) Try starting the arrow string with 6, 7, 8, 9, 10 and explain the pattern.

4b) $6 \xrightarrow{\cdot 3} 18 \xrightarrow{+6} 24 \xrightarrow{:3} 8$

Try starting each arrow string with 4, 5, 6, 7, 8. What do you notice?

5 a) Start 3 $\xrightarrow{\cdot 7}$ 21 $\xrightarrow{+7}$ 28 $\xrightarrow{:7}$ Target 4

b) Start 3 $\xrightarrow{\cdot 7}$ 21 $\xrightarrow{+14}$ 35 $\xrightarrow{:7}$ Target 5

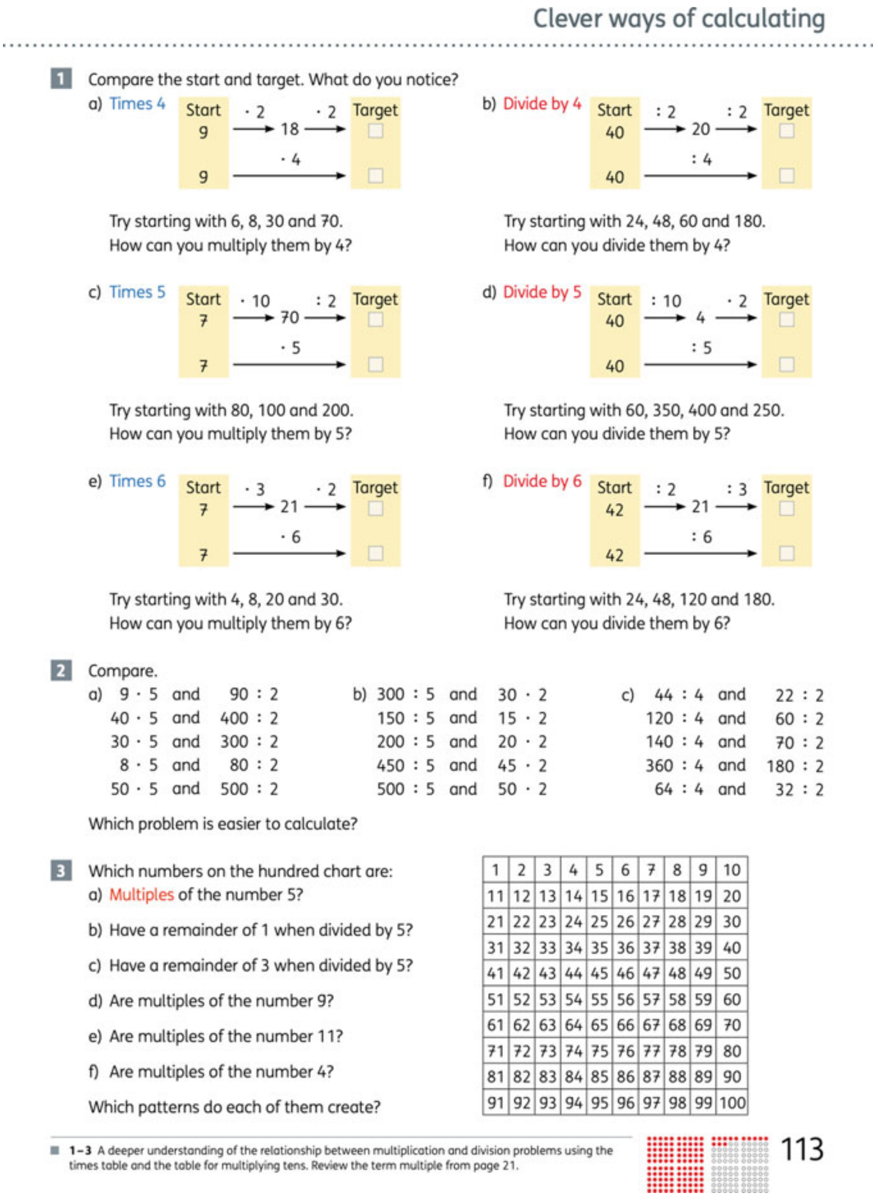
c) Start 3 $\xrightarrow{\cdot 7}$ 21 $\xrightarrow{-7}$ 14 $\xrightarrow{:7}$ Target 2

6 a) Start 3 $\xrightarrow{\cdot 5}$ 15 $\xrightarrow{+5}$ 20 $\xrightarrow{:5}$ Target 4

b) Start 3 $\xrightarrow{\cdot 5}$ 15 $\xrightarrow{+10}$ 25 $\xrightarrow{:5}$ Target 5

c) Start 3 $\xrightarrow{\cdot 5}$ 15 $\xrightarrow{-5}$ 10 $\xrightarrow{:5}$ Target 2

■ **1–6** Review basic exercise formats. Use operations with the angle card and hundred array to explain ("operative proof"). → Workbook, pages 8, 9



Clever ways of calculating

1 Compare the start and target. What do you notice?

a) Times 4

Start 9 $\xrightarrow{\cdot 2}$ 18 $\xrightarrow{\cdot 2}$ Target ☐

9 $\xrightarrow{\cdot 4}$ ☐

Try starting with 6, 8, 30 and 70.
How can you multiply them by 4?

b) Divide by 4

Start 40 $\xrightarrow{:2}$ 20 $\xrightarrow{:2}$ Target ☐

40 $\xrightarrow{:4}$ ☐

Try starting with 24, 48, 60 and 180.
How can you divide them by 4?

c) Times 5

Start 7 $\xrightarrow{\cdot 10}$ 70 $\xrightarrow{:2}$ Target ☐

7 $\xrightarrow{\cdot 5}$ ☐

Try starting with 80, 100 and 200.
How can you multiply them by 5?

d) Divide by 5

Start 40 $\xrightarrow{:10}$ 4 $\xrightarrow{\cdot 2}$ Target ☐

40 $\xrightarrow{:5}$ ☐

Try starting with 60, 350, 400 and 250.
How can you divide them by 5?

e) Times 6

Start 7 $\xrightarrow{\cdot 3}$ 21 $\xrightarrow{\cdot 2}$ Target ☐

7 $\xrightarrow{\cdot 6}$ ☐

Try starting with 4, 8, 20 and 30.
How can you multiply them by 6?

f) Divide by 6

Start 42 $\xrightarrow{:2}$ 21 $\xrightarrow{:3}$ Target ☐

42 $\xrightarrow{:6}$ ☐

Try starting with 24, 48, 120 and 180.
How can you divide them by 6?

2 Compare.

a)	9 · 5 and 90 : 2	b)	300 : 5 and 30 · 2	c)	44 : 4 and 22 : 2		
	40 · 5 and 400 : 2		150 : 5 and 15 · 2		120 : 4 and 60 : 2		
	30 · 5 and 300 : 2		200 : 5 and 20 · 2		140 : 4 and 70 : 2		
	8 · 5 and 80 : 2		450 : 5 and 45 · 2		360 : 4 and 180 : 2		
	50 · 5 and 500 : 2		500 : 5 and 50 · 2		64 : 4 and 32 : 2		

Which problem is easier to calculate?

3 Which numbers on the hundred chart are:

a) Multiples of the number 5?

b) Have a remainder of 1 when divided by 5?

c) Have a remainder of 3 when divided by 5?

d) Are multiples of the number 9?

e) Are multiples of the number 11?

f) Are multiples of the number 4?

Which patterns do each of them create?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

■ **1–3** A deeper understanding of the relationship between multiplication and division problems using the times table and the table for multiplying tens. Review the term multiple from page 21.

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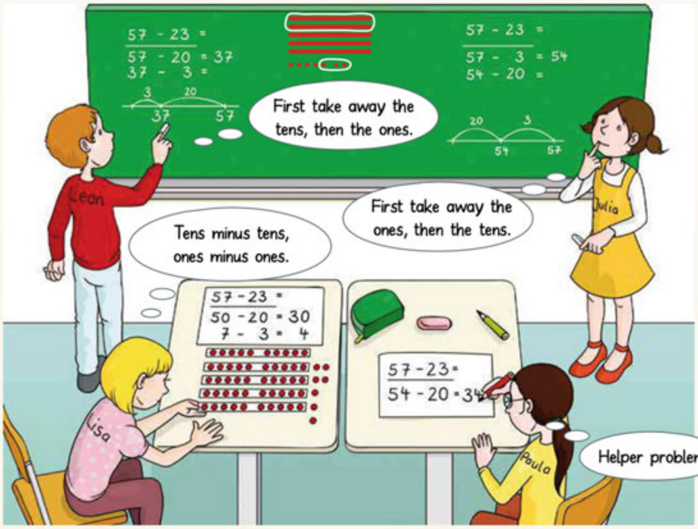
Ways of calculating subtraction problems

1 **How do you calculate 57 – 23?**

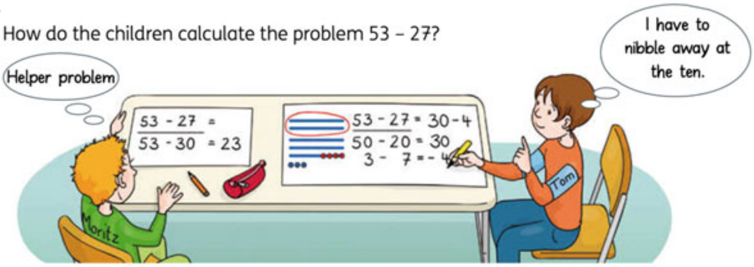


How are the children calculating? Which easy problems do they use?
Compare with how you calculated.

2 Try for yourself.

a) 48 – 25 b) 66 – 41 c) 39 – 27 d) 77 – 64 e) 38 – 15

3 How do the children calculate the problem 53 – 27?

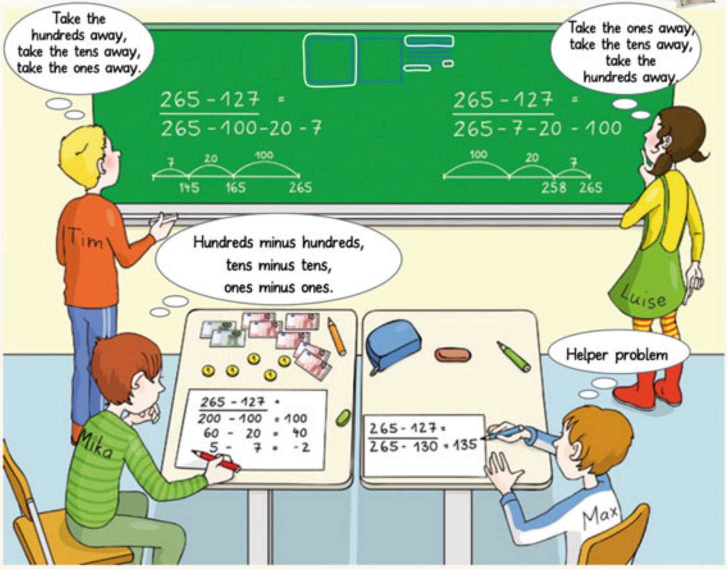


50

1 Calculate 57 – 23 on their own first. Point out various ways of calculating and discuss (maths conference). Encourage children to find their own ways. **2** Find own ways of calculating. **3** Calculate 53 – 27 on their own first; then discuss the children's ways of calculating.

Ways of calculating subtraction problems

1 How do you calculate 265 – 127?

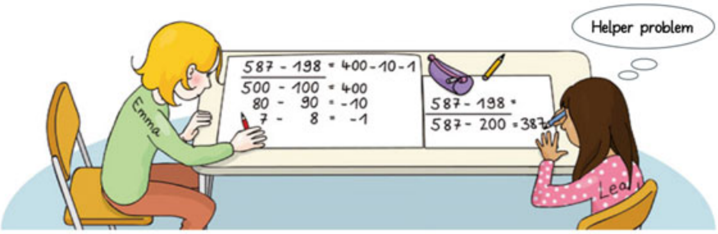


How are the children calculating? Which easy problems do they use?
Compare with how you calculated.

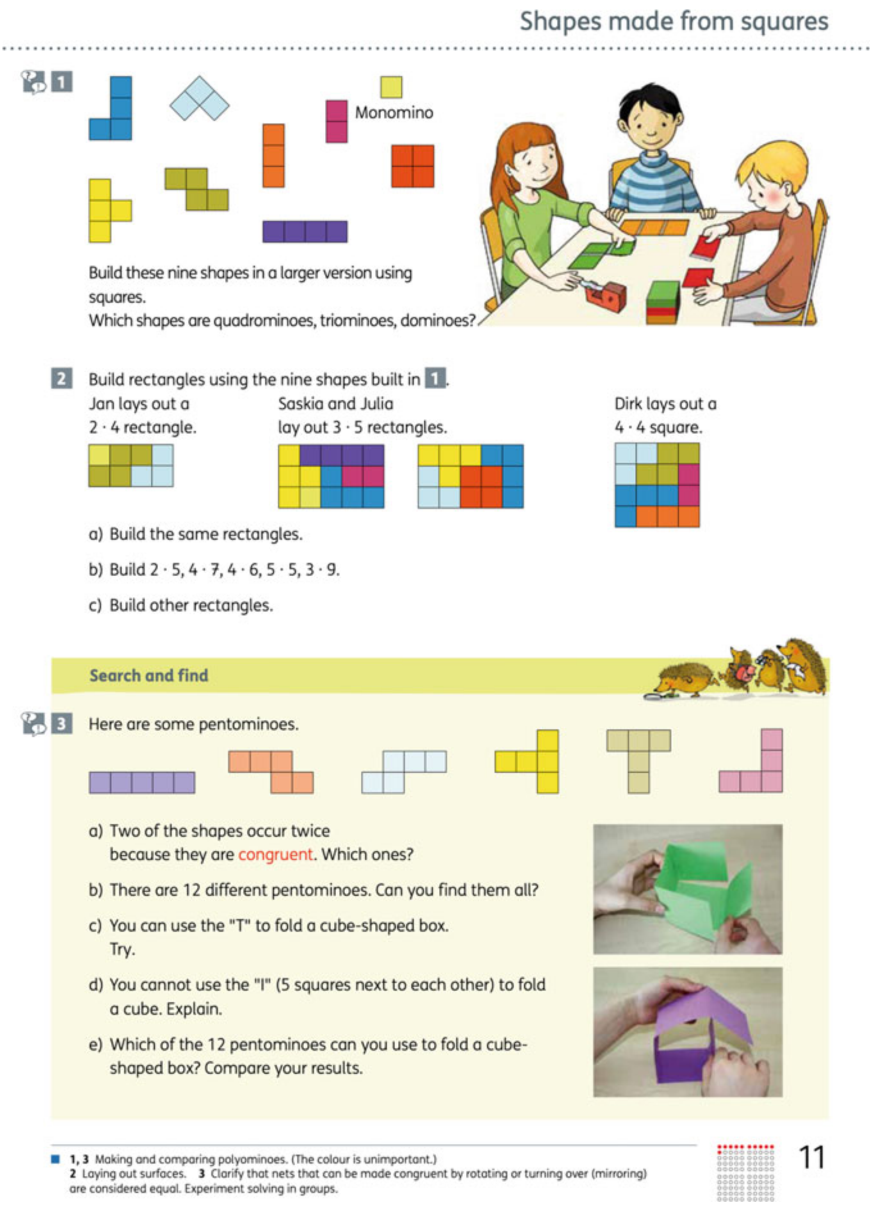
2 Try for yourself. To check your work, calculate in two different ways.

a) 641 – 235 b) 821 – 118 c) 165 – 27 d) 365 – 227 e) 270 – 132

3 How do the children calculate the problem 587 – 198? Compare with your own ways of calculating.



■ **1** Have children calculate 265 – 127 themselves before working with the book. Have children present different ways of calculating, discuss and compare with the ways of calculating indicated (maths conference). Encourage children to find their own ways.
2 Try own ways of calculating. **3** Have children calculate on their own first and then discuss their ways of calculating.



Shapes made from squares

1

Monomino

Build these nine shapes in a larger version using squares.
Which shapes are quadrominoes, triominoes, dominoes?

2 Build rectangles using the nine shapes built in **1**.

Jan lays out a $2 \cdot 4$ rectangle.

Saskia and Julia lay out $3 \cdot 5$ rectangles.

Dirk lays out a $4 \cdot 4$ square.

a) Build the same rectangles.

b) Build $2 \cdot 5$, $4 \cdot 7$, $4 \cdot 6$, $5 \cdot 5$, $3 \cdot 9$.

c) Build other rectangles.

Search and find

3 Here are some pentominoes.

a) Two of the shapes occur twice because they are congruent. Which ones?

b) There are 12 different pentominoes. Can you find them all?

c) You can use the "T" to fold a cube-shaped box. Try.

d) You cannot use the "I" (5 squares next to each other) to fold a cube. Explain.

e) Which of the 12 pentominoes can you use to fold a cube-shaped box? Compare your results.

■ **1, 3** Making and comparing polyominoes. (The colour is unimportant.)
2 Laying out surfaces. **3** Clarify that nets that can be made congruent by rotating or turning over (mirroring) are considered equal. Experiment solving in groups.

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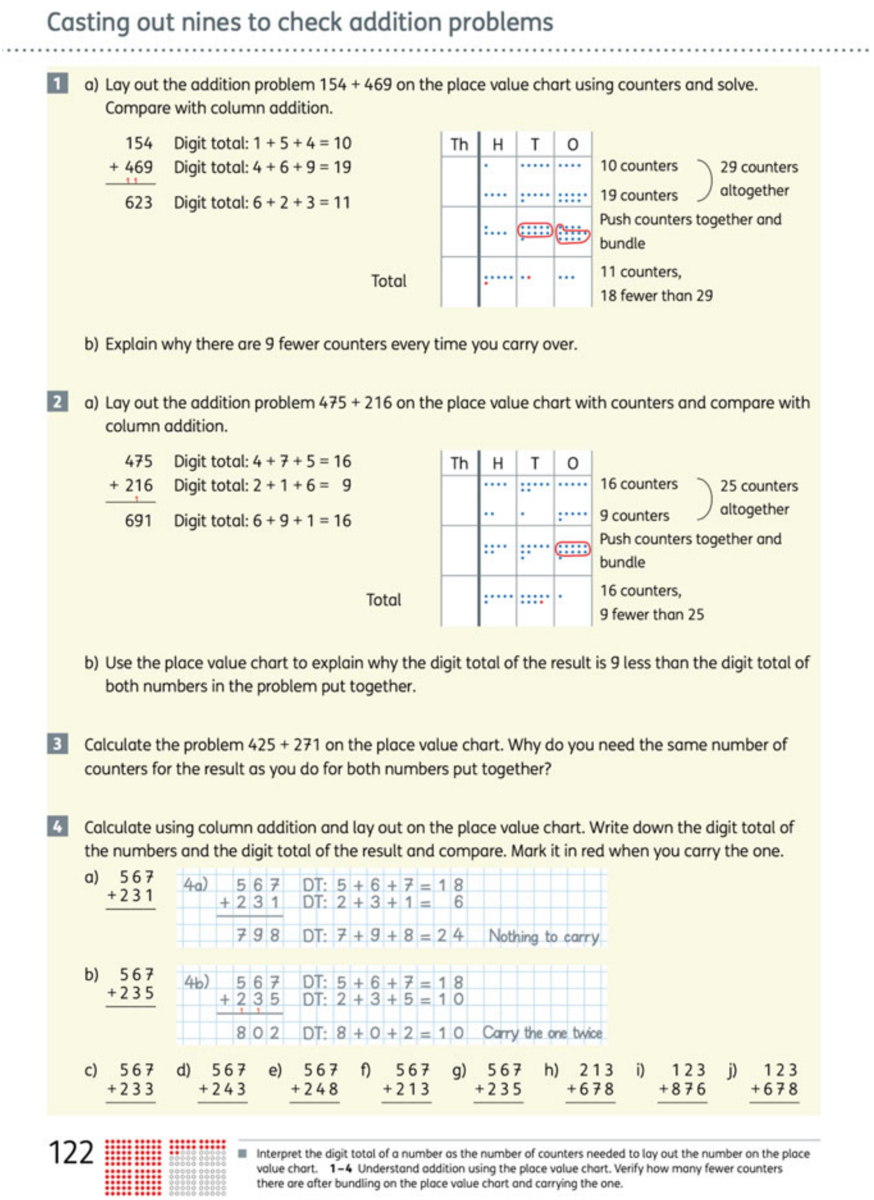

Casting out nines to check addition problems

1 a) Lay out the addition problem 154 + 469 on the place value chart using counters and solve. Compare with column addition.

```
  154    Digit total: 1 + 5 + 4 = 10
+ 469    Digit total: 4 + 6 + 9 = 19
   11
  623    Digit total: 6 + 2 + 3 = 11
```

Th	H	T	O	
				10 counters 19 counters — 29 counters altogether
				Push counters together and bundle
Total				11 counters, 18 fewer than 29

b) Explain why there are 9 fewer counters every time you carry over.

2 a) Lay out the addition problem 475 + 216 on the place value chart with counters and compare with column addition.

```
  475    Digit total: 4 + 7 + 5 = 16
+ 216    Digit total: 2 + 1 + 6 =  9
    1
  691    Digit total: 6 + 9 + 1 = 16
```

Th	H	T	O	
				16 counters 9 counters — 25 counters altogether
				Push counters together and bundle
Total				16 counters, 9 fewer than 25

b) Use the place value chart to explain why the digit total of the result is 9 less than the digit total of both numbers in the problem put together.

3 Calculate the problem 425 + 271 on the place value chart. Why do you need the same number of counters for the result as you do for both numbers put together?

4 Calculate using column addition and lay out on the place value chart. Write down the digit total of the numbers and the digit total of the result and compare. Mark it in red when you carry the one.

a) 567 + 231

```
4a)   5 6 7   DT: 5 + 6 + 7 = 1 8
    + 2 3 1   DT: 2 + 3 + 1 =   6

      7 9 8   DT: 7 + 9 + 8 = 2 4   Nothing to carry
```

b) 567 + 235

```
4b)   5 6 7   DT: 5 + 6 + 7 = 1 8
    + 2 3 5   DT: 2 + 3 + 5 = 1 0
        1 1
      8 0 2   DT: 8 + 0 + 2 = 1 0   Carry the one twice
```

c) 567 + 233 d) 567 + 243 e) 567 + 248 f) 567 + 213 g) 567 + 235 h) 213 + 678 i) 123 + 876 j) 123 + 678

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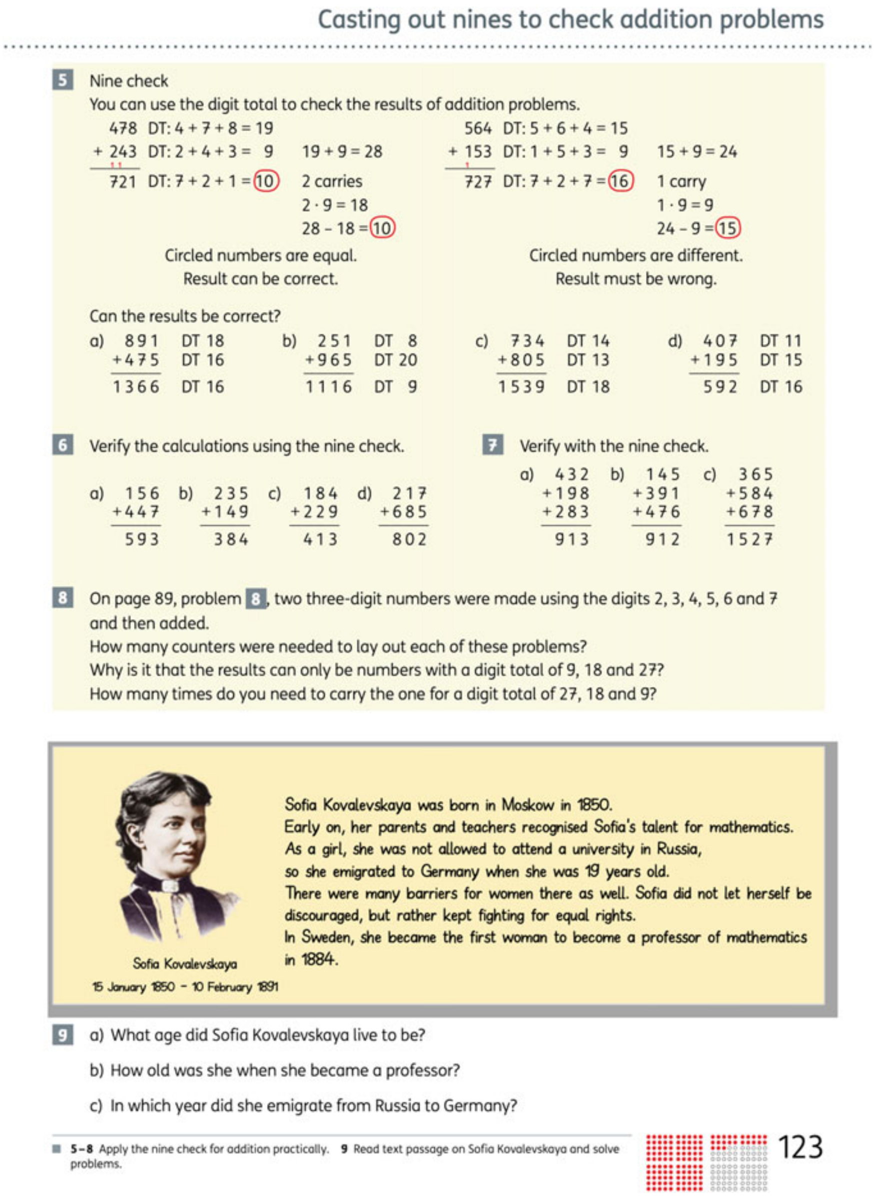
■ Interpret the digit total of a number as the number of counters needed to lay out the number on the place value chart. **1–4** Understand addition using the place value chart. Verify how many fewer counters there are after bundling on the place value chart and carrying the one.



Casting out nines to check addition problems

5 Nine check

You can use the digit total to check the results of addition problems.

478 DT: 4 + 7 + 8 = 19
+ 243 DT: 2 + 4 + 3 = 9 19 + 9 = 28
1 1
721 DT: 7 + 2 + 1 = $\boxed{10}$ 2 carries
2 · 9 = 18
28 – 18 = $\boxed{10}$

Circled numbers are equal.
Result can be correct.

564 DT: 5 + 6 + 4 = 15
+ 153 DT: 1 + 5 + 3 = 9 15 + 9 = 24
1
727 DT: 7 + 2 + 7 = $\boxed{16}$ 1 carry
1 · 9 = 9
24 – 9 = $\boxed{15}$

Circled numbers are different.
Result must be wrong.

Can the results be correct?

a)	891	DT 18	b)	251	DT 8	c)	734	DT 14	d)	407	DT 11
	+475	DT 16		+965	DT 20		+805	DT 13		+195	DT 15
	1366	DT 16		1116	DT 9		1539	DT 18		592	DT 16

6 Verify the calculations using the nine check.

a)	156	b)	235	c)	184	d)	217
	+447		+149		+229		+685
	593		384		413		802

7 Verify with the nine check.

a)	432	b)	145	c)	365
	+198		+391		+584
	+283		+476		+678
	913		912		1527

8 On page 89, problem **8**, two three-digit numbers were made using the digits 2, 3, 4, 5, 6 and 7 and then added.
How many counters were needed to lay out each of these problems?
Why is it that the results can only be numbers with a digit total of 9, 18 and 27?
How many times do you need to carry the one for a digit total of 27, 18 and 9?

Sofia Kovalevskaya
15 January 1850 – 10 February 1891

Sofia Kovalevskaya was born in Moskow in 1850.
Early on, her parents and teachers recognised Sofia's talent for mathematics.
As a girl, she was not allowed to attend a university in Russia,
so she emigrated to Germany when she was 19 years old.
There were many barriers for women there as well. Sofia did not let herself be discouraged, but rather kept fighting for equal rights.
In Sweden, she became the first woman to become a professor of mathematics in 1884.

9 a) What age did Sofia Kovalevskaya live to be?

b) How old was she when she became a professor?

c) In which year did she emigrate from Russia to Germany?

■ **5–8** Apply the nine check for addition practically. **9** Read text passage on Sofia Kovalevskaya and solve problems.

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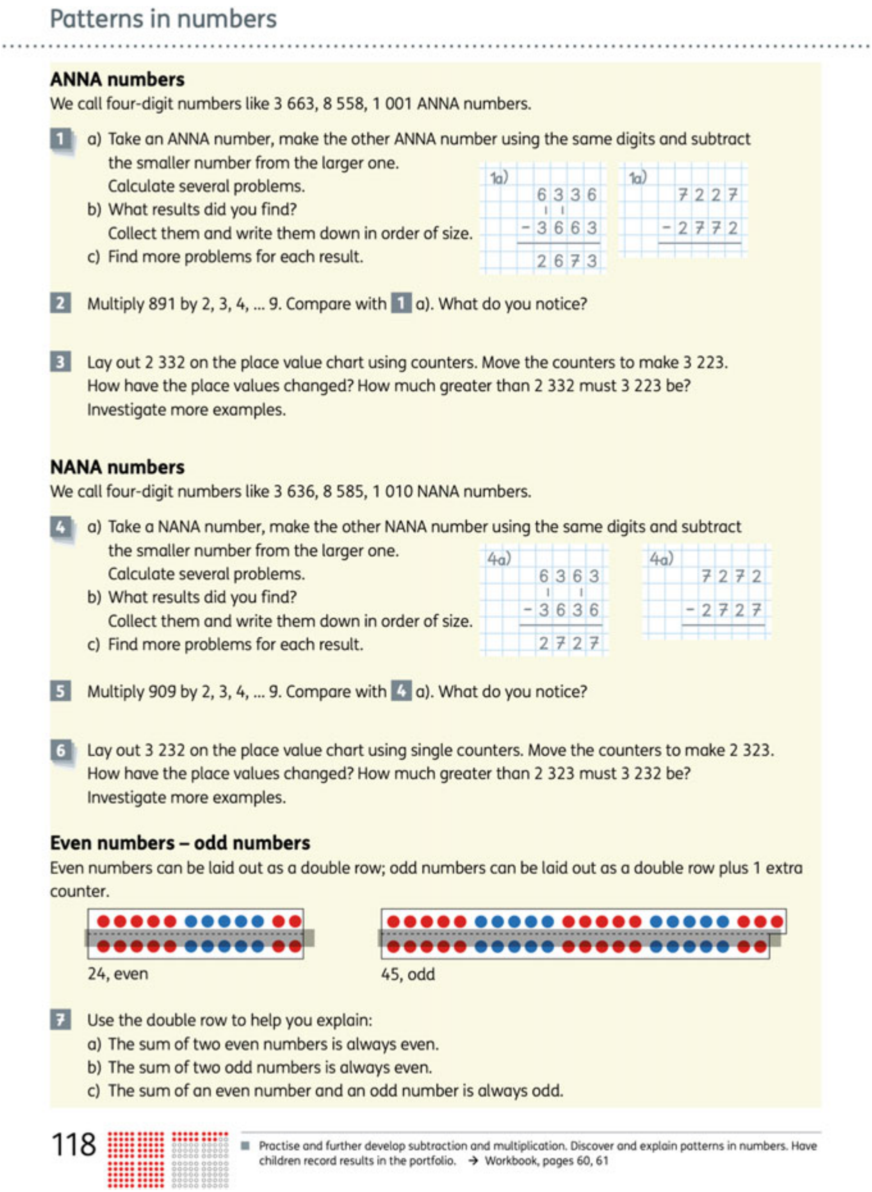

Patterns in numbers

ANNA numbers

We call four-digit numbers like 3 663, 8 558, 1 001 ANNA numbers.

1 a) Take an ANNA number, make the other ANNA number using the same digits and subtract the smaller number from the larger one.
Calculate several problems.
b) What results did you find?
Collect them and write them down in order of size.
c) Find more problems for each result.

1a)
6 3 3 6
− 3 6 6 3
2 6 7 3

1a)
7 2 2 7
− 2 7 7 2

2 Multiply 891 by 2, 3, 4, … 9. Compare with **1** a). What do you notice?

3 Lay out 2 332 on the place value chart using counters. Move the counters to make 3 223.
How have the place values changed? How much greater than 2 332 must 3 223 be?
Investigate more examples.

NANA numbers

We call four-digit numbers like 3 636, 8 585, 1 010 NANA numbers.

4 a) Take a NANA number, make the other NANA number using the same digits and subtract the smaller number from the larger one.
Calculate several problems.
b) What results did you find?
Collect them and write them down in order of size.
c) Find more problems for each result.

4a)
6 3 6 3
− 3 6 3 6
2 7 2 7

4a)
7 2 7 2
− 2 7 2 7

5 Multiply 909 by 2, 3, 4, … 9. Compare with **4** a). What do you notice?

6 Lay out 3 232 on the place value chart using single counters. Move the counters to make 2 323.
How have the place values changed? How much greater than 2 323 must 3 232 be?
Investigate more examples.

Even numbers – odd numbers

Even numbers can be laid out as a double row; odd numbers can be laid out as a double row plus 1 extra counter.

24, even

45, odd

7 Use the double row to help you explain:
a) The sum of two even numbers is always even.
b) The sum of two odd numbers is always even.
c) The sum of an even number and an odd number is always odd.

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Practise and further develop subtraction and multiplication. Discover and explain patterns in numbers. Have children record results in the portfolio. → Workbook, pages 60, 61